A UNIFIED APPROACH TO TWO-TIME-SCALE CONTROL SYSTEMS DESIGN: A TUTORIAL

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ABSTRACT
The goal of the tutorial is to give an overview of the newest unified design methodology of continuous-time or discrete-time nonlinear control systems which guarantees desired transient performances in the presence of plant parameter variations and unknown external disturbances. The tutorial presents the up-to-date coverage of fundamental issues and recent research developments in design of nonlinear control systems with the highest derivative in feedback. The discussed design methodology allows us to provide effective control of nonlinear systems on the assumption of uncertainty. The approach is based on an application of a dynamical control law with the highest derivative of the output signal in the feedback loop. A distinctive feature of the control systems thus designed is that two-time-scale motions are forced in the closed-loop system. Stability conditions imposed on the fast and slow modes, and a sufficiently large mode separation rate, can ensure that the full-order closed-loop system achieves desired properties: the output transient performances are as desired, and they are insensitive to parameter variations and external disturbances. A general design methodology for control systems with the highest derivative in feedback for continuous-time systems, as well as corresponding discrete-time counterpart, will be presented during this tutorial. The method of singular perturbation is used to analyze the closed-loop system properties throughout.

KEY WORDS
Control under uncertainty, singular perturbation method.

1 Introduction

The problem of output regulation of nonlinear time-varying control systems under uncertainties is one of particular interest for real-time control system design. There is a broad set of practical problems in the control of aircraft, robotics, chemical industry, and electro-mechanical systems where control systems are designed to provide: (i) robust zero steady-state error of the reference input realization, (ii) desired output performance specifications such as overshoot, settling time, and system type, (iii) insensitivity of the output transient behavior with respect to unknown external disturbances and varying parameters of the system.

In order to solve the output regulation problem, control system design can efficiently be done under uncertainty via the design methodologies for control systems with sliding motions [1, 2], control systems with high gain in feedback [3, 4], and control systems with the highest-order derivative and high gain in feedback [5, 6, 7]. A distinctive feature of the control systems thus designed is that two-time-scale motions are forced in the closed-loop system. Hence, the method of singular perturbation [8, 9] is used to analyze the closed-loop system properties in such systems. Note that the control with the highest derivative in feedback is related with acceleration feedback control widely used in robotics [10, 11, 12, 13].

The tutorial presents the unified approach to continuous as well as digital control system design. The main advantage of the control systems thus designed is that the desired output transient performances are provided in the presence of plant parameter variations and unknown external disturbances.

The paper is organized as follows. First, some preliminary results concern with properties of singularly perturbed systems and control systems with the highest derivative in feedback are discussed. Second, the application of the discussed design methodology for a simple model of continuous-time single-input single-output nonlinear system is discussed and main steps of the design method are explained. Third, the discrete-time counterpart of the discussed method for sampled-data control system design is presented. The way of extension of the discussed design methodology for general systems will be done during tutorial presentation.

2 Preliminary results

2.1 Continuous-time singularly perturbed systems

Let us consider the following system:

\[
\dot{X} = f(X, Z), \quad X(0) = X^0, \quad (1)
\]

\[
\mu \dot{Z} = g(X, Z), \quad Z(0) = Z^0, \quad (2)
\]

where \( \mu \) is a small positive parameter, \( X \in \mathbb{R}^n \), \( Z \in \mathbb{R}^m \), and \( f \) and \( g \) are continuously differentiable functions of \( X \).
and $Z$. The system (1)–(2) is called the standard singularly perturbed system.

From (1)–(2) we can get the fast motion subsystem (FMS) given by

$$\mu \frac{dZ}{dt} = g(X, Z), \quad Z(0) = Z^0, \quad (3)$$

as $\mu \to 0$ where $X(t)$ is the frozen variable. Assume that $g(X, Z)$ exists such that for all $\mu \in (0, \mu^*)$, the trajectories of the perturbed systems approximate to the trajectories of the singularly perturbed system.

Hence, the function $Z = \psi(X)$ exists such that $g(X(t), Z(t)) = 0 \quad \forall \ t$ holds where $Z$ is an equilibrium point of (3). Assume that the equilibrium point $Z$ is unique and one is stable (exponentially stable). Hence, from (1)–(2), the slow motion subsystem (SMS) (or a so-called reduced system)

$$\dot{X} = f(X, \psi(X)), \quad X(0) = X^0 \quad (5)$$

follows.

### 2.2 Discrete-time singularly perturbed systems

Let us consider the system of difference equations given by

$$X_{k+1} = \{I_n + \mu A_{11}\}X_k + \mu A_{12}Y_k, \quad (6)$$

$$Y_{k+1} = A_{21}X_k + A_{22}Y_k, \quad (7)$$

where $\mu$ is a small parameter, $X \in \mathbb{R}^n$, $Y \in \mathbb{R}^m$, and the $A_{ij}$ are matrices with appropriate dimensions.

If $\mu$ is sufficiently small, then from (6)–(7) the FMS equation

$$Y_{k+1} = A_{21}X_k + A_{22}Y_k \quad (8)$$

results, where $X_k \approx \text{const}$ during the transients in the system (8).

Assume that the FMS (8) is stable. Then the steady-state of the FMS is given by

$$Y_k = \{I_m - A_{22}\}^{-1}A_{21}X_k. \quad (9)$$

Substitution of (9) into (6) yields the SMS

$$X_{k+1} = \{I_n + \mu[A_{11} + A_{12}(I_m - A_{22})^{-1}A_{21}]\}X_k. \quad (10)$$

The main qualitative property of the singularly perturbed systems is that: if the equilibrium point of the FMS is stable (exponentially stable), then there exists $\mu^* > 0$ such that for all $\mu \in (0, \mu^*)$, the trajectories of the singularly perturbed system approximate to the trajectories of the SMS [15, 16, 14]. This property is important both from a theoretical viewpoint and for practical applications in control system analysis and design.

### 2.3 Control problem

Consider a nonlinear system of the form

$$\frac{dx}{dt} = f(x, w) + g(x, w)u, \quad x(0) = x_0, \quad (10)$$

where $t$ denotes time, $t \in [0, \infty)$, $y = x$ is the measurable output of the system (10), $x \in \mathbb{R}^l$, $u$ is the control, $u \in \Omega_u \subset \mathbb{R}^1$, $w$ is the vector of unknown bounded external disturbances or varying parameters, $w \in \Omega_w \subset \mathbb{R}^l$, and $\|w(t)\| \leq w_{\text{max}} < \infty$, $w_{\text{max}} > 0$.

We assume that $dw/dt$ is bounded for all its components,

$$\|dw/dt\| \leq \bar{w}_{\text{max}} < \infty, \quad (11)$$

and that the conditions

$$0 < g_{\text{min}} \leq |g(x, w)| \leq g_{\text{max}} < \infty, \quad (12)$$

are satisfied for all $(x, w) \in \Omega_{x, w}$, where $f(x, w), g(x, w)$ are unknown continuous bounded functions of $x(t), w(t)$ on the bounded set $\Omega_{x, w}$ and $\bar{w}_{\text{max}} > 0$, $g_{\text{min}} > 0$, $g_{\text{max}} > 0, f_{\text{max}} < \infty$.

A control system is being designed so that

$$\lim_{t \to \infty} e(t) = 0, \quad (13)$$

where $e(t)$ is an error of the reference input realization: $e(t) = r(t) - x(t)$ and $r(t)$ is the reference input.

Moreover, the output transients should have the desired performance indices. These transients should not depend on the external disturbances and varying parameters $w(t)$ of the system (10).

### 2.4 Insensitivity condition

Let us construct the reference model for (10) in the form of the 1st order desired stable differential equation

$$\frac{dx}{dt} = F(x, r). \quad (14)$$

For example, let us suppose that (14) is given by

$$\frac{dx}{dt} = \frac{1}{T}(r - x), \quad (15)$$

where $x = r$ at the equilibrium for $r = \text{const}$. Denote

$$e^F = F(x, r) - \frac{dx}{dt}. \quad (16)$$

Accordingly, if the condition

$$e^F = 0 \quad (17)$$

holds, then the behavior of $x(t)$ with prescribed dynamics of (14) is fulfilled. The expression (17) is an insensitivity condition for the behavior of the output $x(t)$ with respect
to the external disturbances and varying parameters of the system (10).

Substitution of (10), (14), and (16) into (17) yields

\[ F(x, r) - f(x, w) - g(x, w)u = 0. \]  

(18)

So, (13) has been reformulated as a problem of finding a solution of the equation \( e^{F(u)} = 0 \) when its varying parameters are unknown. From (18) we get \( u = u^{\text{NID}} \) where

\[ u^{\text{NID}} = \{g(x, w)\}^{-1}\{F(x, r) - f(x, w)\} \]  

(19)

and \( u^{\text{NID}}(t) \) is the analytical solution of (18). The control function \( u(t) = u^{\text{NID}}(t) \) is called a solution of the nonlinear inverse dynamics (NID). It is clear that the control law in the form of (19) may be used only if complete information about the disturbances, model parameters, and state of the system (10).

2.5 Control with the highest derivative in feedback

The subject of our consideration is the problem of control system design given that the functions \( f(X, w), g(X, w) \) are unknown and the vector \( w(t) \) of bounded external disturbances or varying parameters is unavailable for measurement. In order to reach the discussed control goal and, as a result, to provide desired dynamical properties of \( x(t) \) in the specified region of the state space of the uncertain nonlinear system (10), the following control law with the highest derivative of the output signal and high gain in the feedback loop

\[ u = k_0\{F(X, r) - \dot{x}\} \]  

(20)

was proposed [5], where \( k_0 \) is a high gain, \( k_0 \in \mathbb{R}^1 \).

Let us consider the basic correlations of the control system with the highest derivative and high gain in the feedback loop that were discussed in [5, 6, 7]. From the closed-loop system equations

\[ \dot{x} = f(x, w) + g(x, w)u, \]  

(21)

\[ u = k_0\{F(x, r) - \dot{x}\}, \]  

(22)

it follows that

\[ \dot{x} = F(x, r) + \frac{1}{1 + g(x, w)k_0}\{f(X, w) - F(X, r)\} \]  

(23)

and

\[ u = \frac{k_0}{1 + g(X, w)k_0}\{F(X, r) - f(X, w)\}, \]  

(24)

Hence, the conditions

\[ \lim_{gk_0 \to \infty} \dot{x} = F(x, r), \lim_{gk_0 \to \infty} u = u^{\text{NID}} \]  

(25)

hold despite the fact that the functions \( f(X, w), g(X, w) \) are unknown. In other words, the solution to the control problem under consideration is provided by the use of \( \dot{x} \) in the control law and the use of a high gain \( k_0 \).

In order to implement the control law, the estimation \( \dot{x} \) is used; this is received by a real differentiating filter, for instance, of the following form: \( \mu \dot{x} + \dot{x} = x \). As a result of applying the real differentiating filter in order to implement the control law (20) in practice, we get the singularly perturbed closed-loop system equations

\[ \dot{x} = f(x, w) + g(x, w)k_0\{F(x, r) - (x - \dot{x})/\mu\}, \]  

\[ \mu \dot{x} + \dot{x} = x, \]  

where slow and fast motions take place. For the detailed results pertaining to control systems with the highest derivative of the output signal and high gain in the feedback loop, based on an approach known as a “localization method”, we refer the reader to [17].

The design methodology for control systems design with the highest derivative in feedback discussed below allows to get the unified approach to continuous as well as digital control system design, and this will be demonstrated in subsequent sections.

3 Continuous-time control system design

3.1 Continuous-time control law

Let us devote our attention here to implementation of the control law with the highest derivative in the feedback loop, which allows us to incorporate integral action in the control loop without increasing the controller’s order.

First, in order to obtain some justification for the control law structures introduced below, we notice that the condition (17) corresponds to the minimum value of the following unimodal function:

\[ V(u) = 0.5\{e^{F(u)}\}^2. \]  

(26)

Let us consider \( V(u) \) as a Lyapunov function candidate. Then the requirement

\[ \frac{dV(u)}{dt} = \frac{\partial V(u)}{\partial u} \frac{du}{dt} < 0 \]

can be satisfied for all \( \partial V(u)/\partial u \neq 0 \) by the control law in the form

\[ \frac{du}{dt} = -k_0\nabla_u V(u). \]  

(27)

This corresponds to the gradient descent method and, by definition, we have

\[ \nabla_u V(u) = \partial V(u)/\partial u = -g(x, w)e^{F}. \]  

(28)

In accordance with (12), the condition \( \text{sgn}(g(x, w)) = \text{const} \) is satisfied; then, instead of (27), we can use

\[ \frac{du}{dt} = k_0e^{F}. \]  

(29)
It is easy to see that an equilibrium of (29) is the solution of equation (18).

For the next step, as a generalization of (27), let us consider the control law given by

$$
\mu \frac{du}{dt} + d_0 u = -k_0 \nabla_u V(u).
$$

In a fashion analogous to the above, the control law for the system (10) can be introduced by the following equation:

$$
\mu \frac{du}{dt} + d_0 u = k_0 e^F,
$$

(30)

where $\mu$ is a small positive parameter, $k_0$ is a high gain, and $d_0 = 1$, or $d_0 = 0$.

According to (15) and (16), the control law (30) may be rewritten in the following form:

$$
\mu \frac{du}{dt} + d_0 u = k_0 \left\{ \frac{1}{T}(r - x) - \frac{dx}{dt} \right\}.
$$

(31)

For purposes of numerical simulation or implementation, the control law (31) may be presented in a state-space form such as

$$
\frac{du_1}{dt} = -\frac{d_0}{\mu} u_1 + k_0 \left\{ \frac{d_0}{\mu} - \frac{1}{T} \right\} x + \frac{k_0}{T} r,
$$

(32)

$$
u = \frac{1}{\mu} u_1 - \frac{k_0}{\mu} x.
$$

3.2 Two-time-scale motion analysis

In accordance with (10), (16), and (30), the equations of the closed-loop system are given by

$$
\frac{dx}{dt} = f(x, w) + g(x, w)u, \quad x(0) = x^0,
$$

(32)

$$
\mu \frac{du}{dt} = -d_0 u + k_0 \left\{ F(x, r) - \frac{dx}{dt} \right\}, \quad u(0) = u^0,
$$

(33)

where the highest output derivative $x^{(1)}(t)$ of the plant model (10) is used in feedback.

Substitution of (32) into (33) yields the closed-loop system equations in the form

$$
\frac{dx}{dt} = f(x, w) + g(x, w)u, \quad x(0) = x^0,
$$

(34)

$$
\mu \frac{du}{dt} = -\{d_0 + k_0 g(x, w)\} u
$$

$$+ k_0 \{ F(x, r) - f(x, w) \}, \quad u(0) = u^0,
$$

(35)

where $\mu$ is a small positive parameter.

Since $\mu$ is small, from (34)-(35), we obtain the FMS given by

$$
\mu \frac{du}{dt} + \{d_0 + k_0 g(x, w)\} u = k_0 \{ F(x, r) - f(x, w) \}
$$

(36)

where $x(t)$ and $w(t)$ are frozen variables during the transients in (36).

Let the FMS is stable, then after the rapid decay of transients in (36), we have the steady state (more precisely, quasi-steady state) for the FMS (36), where $u(t) = u^*(t)$ and

$$
u^* = \frac{k_0}{d_0 + k_0 g(x, w)} \{ f(x, r) - F(x, r) \}.
$$

(37)

From (19) and (37) it follows that

$$
u^* = u^N + \frac{d_0}{d_0 + k_0 g(x, w)} \{ f(x, w) - F(x, r) \}
$$

(38)

If the steady state of the FMS (36) takes place, then the closed-loop system equations imply that

$$
\frac{dx}{dt} = f(x, r) + \frac{d_0}{d_0 + k_0 g(x, w)} \{ f(x, w) - F(x, r) \}
$$

(38)

is the equation of the SMS, where (38) is an equation with nonvanishing perturbation and the desired equation (14) plays the role of the nominal system.

As a result of (38), if $d_0 = 1$ and $|k_0| \to \infty$, then the transients of $x(t)$ in the SMS are close to the transients of the reference model (14). If $d_0 = 0$, then the SMS (38) is the same as the reference model (14) and the desired output behavior is fulfilled. Moreover, in this case the integral action is incorporated into the control loop without increasing the controller’s order and, accordingly, the robust zero steady-state error is maintained.

Generalization of (31) in the form of higher order differential equation and set of rules for control law parameters selection (e.g., selection of the parameter $\mu$) based on the requirements for degree of time-scale separation between slow and fast motions, control accuracy, and high-frequency sensor noise attenuation have been presented in [22, 23] where the discussed approach has been extended for MIMO systems given by

$$
\dot{X} = f(X, w) + G(X, w)u, \quad X(0) = X^0,
$$

$$
y = h(X, w),
$$

where $X \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, $u \in \mathbb{R}^m$, and $p \leq m \leq n$.

4 Sampled-data control system design

4.1 Control problem and insensitivity condition

In this section the discrete-time counterpart of the above design methodology is discussed.

Let us consider the backward approximation of the nonlinear system (10) preceded by a zero-order hold (ZOH) with the sampling period $T_s$, that is

$$
x_k = x_{k-1} + T_s [f(x_{k-1}, w_{k-1}) + g(x_{k-1}, w_{k-1})u_{k-1}],
$$

(39)

where $x_k$, $w_k$, and $u_k$ represent samples of $x(t)$, $w(t)$, and $u(t)$ at $t = kT_s$, respectively.
The objective is to design a control system having
\[
\lim_{k \to \infty} e_k = 0. \quad (40)
\]
Here \(e_k = r_k - x_k\) is the error of the reference input realization, \(r_k\) being the samples of the reference input \(r(t)\), where the control transients \(e_k \to 0\) should meet the desired performance specifications given by (15).

By a \(Z\)-transform of (15) preceded by a ZOH, the desired pulse transfer function
\[
H_{x_0}(z) = \frac{z - 1}{z} \mathcal{Z}^{-1} \left\{ s^{-1} \left[ \frac{1/T}{s + 1/T} \right] \right\}_{t = kT_s} \]
\[
= \frac{1 - e^{-T_s/T}}{z - e^{-T_s/T}} \quad (41)
\]
follows. Hence, from (41), the desired stable difference equation
\[
x_k = x_{k-1} + T_s a(T_s) [r_k - x_{k-1}] \quad (42)
\]
results, where
\[
a(T_s) = \frac{1 - e^{-T_s/T}}{T_s}, \quad \lim_{T_s \to 0} a(T_s) = 1, \quad (43)
\]
and the output response of (42) corresponds to the assigned output transient performance indices.

Let us rewrite, for short, the desired difference equation (42) as
\[
x_k = F(x_{k-1}, r_{k-1}), \quad (44)
\]
where we have \(r_k = x_k\) at the equilibrium of (44) for \(r_k = \text{const}, \forall k\). Denote
\[
e_k^F = F(x_{k-1}, r_{k-1}) - x_k, \quad (45)
\]
where \(e_k^F\) is the realization error of the desired dynamics assigned by (44). Accordingly, if for all \(k = 0, 1, \ldots\) the condition
\[
e_k^F = 0 \quad (46)
\]
holds, then the desired behavior of \(x_k\) with the prescribed dynamics of (44) is fulfilled. The expression (46) is the insensitivity condition for the output transient performance with respect to the external disturbances and varying parameters of the plant model given by (10). In other words, the control design problem (40) has been reformulated as the requirement (46). The insensitivity condition given by (46) is the discrete-time counterpart of (17) which was introduced for the continuous-time system (10).

### 4.2 Discrete-time control law

Let us consider the following control law:
\[
u_k = u_{k-1} + \lambda(T_s) [F(x_{k-1}, r_{k-1}) - x_k], \quad (47)
\]
where \(\lambda(T_s) = T_s^{-1} \lambda\) and the reference model of the desired output behavior is given by (42).

In accordance with (42) and (45), the control law (47) can be rewritten as the difference equation
\[
u_k = u_{k-1} + \lambda \left\{ a(T_s) [r_{k-1} - x_{k-1}] - \frac{x_k - x_{k-1}}{T_s} \right\}, \quad (48)
\]
The control law (48) is the discrete-time counterpart of the continuous-time control law (27). Note that, for the cost function \(V(u) = |e_k^F|^2\), equation (47) is related to the well known gradient descent method.

### 4.3 Two-time-scale motion analysis

Denote, for short, \(f_{k-1} = f(x_{k-1}, w_{k-1})\) and \(g_{k-1} = g(x_{k-1}, w_{k-1})\). Hence, the closed-loop system equations have the following form:
\[
x_k = x_{k-1} + T_s [f_{k-1} + g_{k-1} u_{k-1}], \quad (49)
\]
\[
u_k = u_{k-1} + \lambda \left\{ a(T_s) [r_{k-1} - x_{k-1}] - \frac{x_k - x_{k-1}}{T_s} \right\}, \quad (50)
\]
Substitution of (49) into (50) yields
\[
x_k = x_{k-1} + T_s [f_{k-1} + g_{k-1} u_{k-1}], \quad (51)
\]
\[
u_k = [1 - \tilde{\lambda} g] u_{k-1} + \tilde{\lambda} \left\{ a(T_s) [r_{k-1} - x_{k-1}] - f_{k-1} \right\}, \quad (52)
\]
First, note that the stability and the rate of the transients of \(u_k\) in (51)–(52) depend on the controller parameter \(\lambda\). Second, \(x_k - x_{k-1} \to 0\) as \(T_s \to 0\). Hence, we have a slow rate of the transients of \(x_k\) as \(T_s \to 0\). Thus, if \(T_s\) is sufficiently small, the two-time-scale transients are induced in the closed-loop system (51)–(52), where the FMS is governed by
\[
u_k = [1 - \tilde{\lambda} g] u_{k-1} + \tilde{\lambda} \left\{ a(T_s) [r_{k-1} - x_{k-1}] - f_{k-1} \right\}, \quad (53)
\]
and \(x_k = x_{k-1}\), i.e., \(x_k = \text{const} (x_k\ is\ the\ frozen\ variable)\) during the transients in the FMS (53).

From (53), the FMS characteristic polynomial
\[
z - 1 + \tilde{\lambda} g \quad (54)
\]
results, where its root lies inside the unit disk (hence, the FMS is stable) if \(0 < \tilde{\lambda} < 2/g\). To ensure stability and fastest transient processes of \(u_k\), let us take the controller parameter \(\lambda = 1/g\), then the root of (54) is placed at the origin. Hence, the deadbeat response of the FMS (53) is provided.

Third, assume that the FMS (53) is stable and consider its steady state (quasi-steady state), i.e.,
\[
u_k - u_{k-1} = 0. \quad (55)
\]
Then, from (53) and (55), we get \(u_k = u_k^*\), where
\[
u_k^* = g^{-1} \left\{ a(T_s) [r_{k-1} - x_{k-1}] - f_{k-1} \right\}. \quad (56)
\]
Substitution of (55) and (56) into (51) yields the SMS of (51)–(52), which is the same as the desired difference equation (42).
If the degree of time-scale separation between fast and slow motions in the closed-loop system (51)–(52) is sufficiently large and the FMS transients are stable, then after the fast transients have vanished, the behavior of $x_k$ tends to the solution of the reference model (42). Accordingly, the controlled output transient process meets the desired performance specifications.

5 Conclusion

In accordance with the presented above approach the fast motions occur in the closed-loop system such that after fast ending of the fast-motion transients, the behavior of the overall singularly perturbed closed-loop system approaches that of the SMS, which is the same as the reference model. The advantage of the method is that an exact knowledge of parameters of the system is not needed. The design methodology may be useful for real-time control system design under uncertainties [18, 19, 20, 21]. A brief description of the design methodology has been done here. The more detailed results as well as numerical examples and simulation analysis can be found in [22, 23].

References