Parabolic Distributed Parameter Control Systems
with Two-Time-Scale Motions

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Abstract—Control problem for a parabolic partial differential equation system is discussed. Control problem is to provide desired transient performances in presence of an unknown external distributed disturbance and time-varying parameters. The presented design methodology of continuous-time distributed controller as well as discrete-time distributed controller guarantees desired transients by inducing of two-time-scale motions in the closed-loop system. Stability conditions imposed on the fast and slow modes and sufficiently large mode separation rate between fast and slow modes can ensure that the full-order closed-loop partial differential equation system achieves the desired properties in such a way that the transient performances are desired and insensitive to external distributed disturbance and parameter variations. The method of singular perturbations is used through the paper.

I. INTRODUCTION

There is a broad set of technical systems, e.g., reaction-diffusion system, chemical tubular reactor, heating for rolling and others where plant model has a form of parabolic partial differential equation (PDE) systems. Various techniques are widely used in order to controller design for distributed parameter systems, such as: pole assignment [5], [17], [19], [20], [28]; optimization technique [2], [4], [8], [16], [27]; adaptive approach [6], [10], [11], [21]; Lyapunov’s direct method [1], [9]. The geometric approach to the problem of output regulation for distributed parameter systems was discussed as well [3].

In particular, PDE control systems with discontinuous feedback [15], [23] and control systems with the highest derivative in feedback [25], [26], [29], [30] are very powerful tools for control system design under uncertainties. A distinctive feature of the control systems thus designed is that two-time-scale motions (TTSM) are forced in the closed-loop system. Hence, the method of singular perturbations [7], [12], [13], [14], [18], [24] is used to analyze the closed-loop system properties in such systems.

This paper proposes a methodology for the synthesis of feedback controller for PDE systems. The design methodology is based on the construction of two-time scale motions in the closed-loop system and one allows to provide desired transients performances for parabolic PDE systems in presence of an unknown external distributed disturbances and time-varying parameters. The paper is a continuation of [29], [30], [31], [32] and is organized as follows. First, a model of the parabolic PDE system is defined. Second, the main steps of the proposed design methodology of continuous-time distributed controller in the form of appropriate PDE system is presented. Third, based on a modal representation of PDE system by Fourier series, the original distributed parameter system governed by controller in the form of PDE system is decomposed into a continuous-time infinite-dimensional control system with infinite-dimensional controller. After that, the design methodology of control systems with the highest time derivative in feedback [32] is applied to each time mode separately. The discussed controller corresponds to the structure of conventional proportional-integral (PI) controller and this will be shown below. Finally, a discrete-time counterpart of the discussed design methodology is presented too.

II. CONTROL PROBLEM FOR PDE SYSTEM

A. One-dimensional parabolic PDE system

Let us consider a heating or diffusion process described by a one-dimensional parabolic PDE system given by

$$\frac{\partial^2 x}{\partial t^2}(z,t) = \frac{\partial^2 x}{\partial z^2}(z,t) + c(t)x(z,t) + w(z,t) + b(t)u(z,t)$$

with the Neumann boundary conditions,

$$\frac{\partial x}{\partial z}(0,t) = \frac{\partial x}{\partial z}(1,t) = 0$$

and initial condition

$$x(z,0) = x^0(z), \quad 0 \leq z \leq 1,$$

where $t$ is the time, $t \in [0,\infty)$, $z$ is the spatial coordinate, $x(z,t)$ is the state, $c(t)$ and $b(t)$ are unknown varying parameters, $|c(t)| \leq c_0 < \infty$, $|d\sigma/dt| \leq c_m < \infty$, $0 < b_{\min} \leq b(t) \leq b_{\max} < \infty$, $|db/dt| \leq b_n < \infty$, $w(z,t)$ is an external distributed disturbance unavailable for measurement, $u(z,t)$ is the distributed control.

Let the spatial distribution $x(z,t)$ is available for measurement. We assume also that for all functions $x(z,t)$, $x^0(z)$, $w(z,t)$, and $u(z,t)$ the eigenfunction expansions

$$x(z,t) = \sum_{n=0}^{\infty} x_n(t)\varphi_n(z), \quad x^0(z) = \sum_{n=0}^{\infty} x^0_n\varphi_n(z),$$

$$u(z,t) = \sum_{n=0}^{\infty} u_n(t)\varphi_n(z), \quad w(z,t) = \sum_{n=0}^{\infty} w_n(t)\varphi_n(z),$$

hold, where $\varphi_n(z) = \varphi_0^0 \cos(\pi nz)$ are eigenfunctions (also known as spatial modes or normal modes), $\pi^2 n^2$ are eigenvalues, $\varphi_0^0 = 1$, $\varphi_n^0 = \sqrt{2}$, $\forall n = 1, 2, \ldots$.

In this case, the time functions $x_n(t)$ for $x(z,t)$ satisfy the following equations [17], [28]:

$$\ddot{x}_n(t) = [c(t) - \pi^2 n^2]x_n(t) + w_n(t) + b(t)u_n(t),$$

$$x_n(0) = x^0_n, \quad n = 0, 1, \ldots.$$
From (4), we get that some or all of the first \( n_0 \) equations can be unstable due to variations of the parameter \( c(t) \), where \( n_0 = \text{int}(\sqrt{c_0}/\pi) \). Here \( \text{int}(y) \) is the integer part of \( y \). Therefore, the first \( n_0 \) modes should be controlled in order to guarantee close-loop system stability.

**B. Control problem statement**

Let us introduce a desired spatial distribution (reference input) assigned by a function \( x^d(z, t) \) where the boundary conditions of \( x^d(z, t) \) are the same as (2) and the eigenfunction expansion

\[
x^d(z, t) = \sum_{n=0}^{\infty} x^d_n(t) \varphi_n(z)
\]

holds. Denote

\[
e(z, t) = x^d(z, t) - x(z, t)
\]

where \( e(z, t) \) is an error of the desired spatial distribution realization. The discussed control problem consists of the following requirements:

(i) Provide the desired spatial distribution assigned by the function \( x^d(z, t) \), that is:

\[
\lim_{t \to \infty} \sup_{0 \leq z \leq 1} e(z, t) = 0.
\]

(ii) The control transients \( e(z, t) \to 0 \) should have desired transient performance indices. These transients should not depend on the unknown external distributed disturbance \( w(z, t) \) and varying parameters \( c(t), b(t) \) of the system (1).

(iii) Since the control resource is always bounded in a real PDE control system, then assume that the region of allowable values of the distributed control function \( u(z, t) \) is bounded by

\[
|u(z, t)| \leq M_u < \infty, \quad \forall \, t \geq 0 \quad \text{and} \quad 0 \leq z \leq 1
\]

where \( M_u = \text{const} \) is such a bound.

**C. Desired PDE system**

In order to satisfy the requirements for desired behavior of \( x(z, t) \) as well as the desired transient performance indices in the partial differential equation (PDE) system given by (1)–(3), let us introduce a reference equation assigned by the following desired PDE system:

\[
\frac{\partial x}{\partial t}(z, t) = F(x(z, t), x^d(z, t)),
\]

where \( F \) can be selected in the form

\[
\frac{\partial x}{\partial t}(z, t) = \frac{1}{T} [x^d(z, t) - x(z, t)]
\]

with the same boundary conditions and initial condition as (2), (3). From (9) we get \( x^d(z, t) = x(z, t) \) if the steady state of (9) holds. Note that the reference model can be constructed based on some other appropriate PDE system, e.g.,

\[
\frac{\partial x}{\partial t}(z, t) = \frac{1}{T} \{ \frac{\partial^2 [x - x^d]}{\partial z^2}(z, t) + c[x^d(z, t) - x(z, t)] \}
\]

with the boundary conditions and initial condition given by (2), (3). Here \( c > 0, \ T > 0 \) and \( \bar{c}, \ T \) are selected in accordance with the desired settling time in (8).

**D. Insensitivity condition**

Denote

\[
e^F(z, t) = F(x(z, t), x^d(z, t)) - \frac{\partial x}{\partial t}(z, t) \tag{11}
\]

where \( e^F(z, t) \) is the deviation of \( \partial x(z, t)/\partial t \) from \( F(x(z, t), x^d(z, t)) \). The value \( e^F(z, t) \) is called the error of the desired dynamics realization which is assigned by equation (8). As a result, we get the solution of the stated above control problem if

\[
e^F(z, t) = 0, \quad \forall \, z \in [0, 1], \quad \forall \, t \in [0, \infty) \tag{12}
\]

holds, where (12) is the insensitivity condition of the transients in the system (1) with respect to the external disturbance \( w(z, t) \), varying parameters \( c(t), b(t) \), and inherent properties of the PDE system (1).

**E. Realizability of desired behavior in PDE system**

From (1), (11), and (12), we get

\[
u^{id}(z, t) = b^{-1}(t) \left[ F(x(z, t), x^d(t)) - \frac{\partial^2 x}{\partial z^2}(z, t) - c(t)x(z, t) - w(z, t) \right], \tag{13}
\]

where \( u^{id}(z, t) \) is the unique solution of (12). Let us call the distributed control function \( u^{id}(z, t) \) as the solution of the inverse dynamics (id) for PDE system given by (1)–(3).

From (13), we get that the stated above control problem solution exists if the inequality

\[
|u^{id}(z, t)| \leq M_u < \infty, \quad \forall \, t \in [0, \infty), \quad \forall \, z \in [0, 1) \tag{14}
\]

holds, where (14) is the realizability condition of desired behavior in the PDE system given by (1)–(3).

**III. PDE CONTROL SYSTEM WITH TTSM**

**A. Control law and closed-loop PDE system**

The distributed control function \( u^{id}(z, t) \) given by (13) can not be directly used as a controller in presence of assigned above uncertainties in (1). Hence, let us consider an approach that allows us to satisfy the requirement (12) on the condition of unknown information about varying parameters of the PDE system and external distributed disturbance.

Let us propose to use a dynamical distributed controller given by the following PDE system:

\[
\frac{\partial u}{\partial t}(z, t) + d_0 u(z, t) = k \left[ F(x(z, t), x^d(z, t)) - \frac{\partial x}{\partial t}(z, t) \right] \tag{15}
\]

where the first-order time derivative of the state \( x(z, t) \) is used in feedback, \( \mu \) is a small positive parameter, \( k \) is a positive gain, and \( d_0 = 0 \) or \( d_0 = 1 \). Note that the boundary conditions and initial condition for (15) can be assigned arbitrarily, in general.
From (1) and (15), the closed-loop PDE system
\[
\frac{\partial x}{\partial t}(z, t) = \frac{\partial^2 x}{\partial z^2}(z, t) + c(t)x(z, t) + w(z, t) + b(t)u(z, t),
\]
\[
\frac{\mu}{\partial t}(z, t) = -d_0u(z, t)
\]
\[
+ k \left[ F(x(z, t), x'(z, t)) - \frac{\partial x}{\partial t}(z, t) \right]
\]
follows. Next, let us consider the closed-loop PDE system properties. Substitution of (16) into (17) yields the closed-loop PDE system in the form of singularly perturbed PDE system
\[
\frac{\partial x}{\partial t}(z, t) = \frac{\partial^2 x}{\partial z^2}(z, t) + c(t)x(z, t) + w(z, t) + b(t)u(z, t),
\]
\[
\frac{\mu}{\partial t}(z, t) = -[d_0 + kb(t)]u(z, t)
\]
\[
+ k \left[ F(x(z, t), x'(z, t)) - \frac{\partial^2 x}{\partial z^2}(z, t) - c(t)x(z, t) - w(z, t) \right].
\]
Since \( \mu \) is the small parameter, the closed-loop PDE system (18)–(19) is the singularly perturbed partial differential equations. If \( \mu \to 0 \), then fast and slow modes are induced in the closed-loop PDE system and the time-scale separation between these modes depends on the controller parameter \( \mu \). Hence, the closed-loop PDE system properties can be analyzed on basis of the singular perturbation technique and, as a result, slow and fast motion PDE subsystems have to be derived and then analyzed separately.

B. Fast-motion PDE subsystem

Introducing the new fast time scale \( t_0 = t/\mu \) into the closed-loop PDE system (18)–(19), we get
\[
\frac{\partial x}{\partial \tau} = \mu \left[ \frac{\partial^2 x}{\partial z^2} + cx + w + bu \right],
\]
\[
\frac{\partial u}{\partial \tau} = -[d_0 + kb(t)]u
\]
\[
+ k \left[ F(x, x') - \frac{\partial^2 x}{\partial z^2} - cx - w \right]
\]
as the closed-loop PDE system in the new time scale \( \tau \). Then, from (20)–(21), we get the fast-motion PDE subsystem in the time scale \( \tau \), that is
\[
\frac{\partial u}{\partial \tau} = -[d_0 + kb(t)]u
\]
\[
+ k \left[ F(x, x') - \frac{\partial^2 x}{\partial z^2} - cx - w \right]
\]
as \( \mu \to 0 \). Then, returning to the primary time scale \( t = \mu \tau \), we obtain the following fast-motion PDE subsystem:
\[
\frac{\partial u}{\partial t}(z, t) = -[d_0 + kb(t)]u(z, t)
\]
\[
+ k \left[ F(x(z, t), x'(z, t)) - \frac{\partial^2 x}{\partial z^2}(z, t) - c(t)x(z, t) - w(z, t) \right],
\]
where the state \( x(z, t) \), the reference input \( x'(z, t) \), the distributed external disturbance \( w(z, t) \), and varying parameters \( c(t), b(t) \) are frozen variables during the transients in (23) if the parameter \( \mu \) is small enough.

By setting the small parameter \( \mu = 0 \) in (23), we get a quasi-steady state mode of the fast-motion PDE subsystem
\[
u^*(z, t) = \left[ d_0 + kb(t)^{-1}k \left[ F(x(z, t), x'(z, t)) - \frac{\partial^2 x}{\partial z^2}(z, t) - c(t)x(z, t) - w(z, t) \right] \right],
\]
where \( u^*(z, t) \) is an asymptotically stable quasi-equilibrium point of the fast-motion PDE subsystem (23) if \( d_0 + kb > 0 \) holds. From (13) and (24) it follows that
\[
u^*(z, t) = u'(z, t) + \left[ d_0 + kb \left[ \frac{\partial^2 x}{\partial z^2}(z, t) + c(t)x(z, t) + w(z, t) - F(x(z, t), x'(z, t)) \right] \right].
\]

C. Slow-motion PDE subsystem

By letting \( \mu = 0 \) in the closed-loop PDE system (18)–(19) (or, substitution of (25) into (1)), we find that
\[
\frac{\partial x}{\partial \tau}(z, t) = F(x(z, t), x'(z, t))
\]
\[
+ \frac{d_0}{d_0 + kb} \left[ \frac{\partial^2 x}{\partial z^2}(z, t) + c(t)x(z, t) + w(z, t) - F(x(z, t), x'(z, t)) \right]
\]
describes the slow-motion PDE subsystem.

Note that if \( d_0 = 0 \) then from (26) we obtain
\[
\lim_{k \to \infty} \frac{\partial x}{\partial \tau}(z, t, k) = F(x(z, t), x'(z, t)),
\]
i.e., \( \lim_{k \to \infty} e^F(z, t, k) = 0 \). In other words, the slow-motion PDE subsystem (26), tends to (8) if the high gain \( k \) is used. On the other hand, if \( d_0 = 0 \) then from (26) we get (8). In contrast to the previous case, the slow-motion PDE subsystem is the same as the desired PDE system (8) even if \( kb \gg 1 \). In this case the integral action is incorporated in the control loop without increasing the controller’s order and, accordingly, the robust zero steady-state error of the desired spatial distribution is maintained in the slow-motion PDE subsystem despite the fact that there are unknown parameters \( c, b \) and unknown external distributed disturbance \( w \).

Accordingly, the desired transients are guaranteed in the closed-loop PDE system after fast damping of fast-motion transients. So, if a sufficient time-scale separation between the fast and slow modes in the closed-loop PDE system and stability of the fast-motion transients are provided, then we get the discussed control problem solution.

IV. FOURIER SERIES OF PDE CONTROL SYSTEM WITH TTSM

The justification and generalization of the above results can be done based on a modal representation by a Fourier series of the discussed PDE control system with TTSM and this will be shown below.
A. Insensitivity condition

Denote by \( e_n(t) = x_n^d(t) - x_n(t) \) a realization error of the desired time function \( x_n(t) \). Then the requirement (6) corresponds to

\[
\lim_{t \to \infty} e_n(t) = 0, \quad n = 0, 1, \ldots
\]  

(27)

So, the desired behavior of \( x(z, t) \) can be provided if the process \( x_n(t) \to x_n^d(t) \) satisfies the desired differential equation

\[
\dot{x}_n(t) = F_n(x_n(t), x_n^d(t))
\]

(28)

for each time function \( x_n(t) \). The parameters of (28) are selected in accordance with assigned transient performance indices and in such a way that the condition \( x_n = x_n^d \) holds for the steady state of (28). For instance, the linear differential equation in the form

\[
\dot{x}_n(t) = T_n^{-1}[x_n^d(t) - x_n(t)]
\]

(29)

is the most convenient in this case where \( T_n \) is selected in accordance with the desired settling time of the transients in (29).

Denote \( e_n^F = F_n - \dot{x}_n(t) \), where \( e_n^F \) is the realization error of the desired dynamics assigned by (28). As a result, we get the solution of the stated above control problem (6) with desired transient performance indices if the requirements

\[
e_n^F = 0, \quad \forall \ n = 0, 1, \ldots
\]

hold, where (30) is the insensitivity condition of the transients in the system (1) with respect to the external disturbance \( w(z, t) \) and varying parameters \( c(t), b(t) \).

By (4) and (28)–(29) the expression (30) can be rewritten in the form

\[
x_n^d(t) - x_n(t)\bigg|_{T_n} + [\pi^2 n^2 - c(t)]x_n(t)
\]

\[
- w_n(t) - b(t) u_n(t) = 0, \quad \forall \ n = 0, 1, \ldots
\]

(31)

So, the discussed control problem has been reformulated as the requirement to provide the condition (31) or, in other words, to find a solution to (31) when its varying parameters are unknown.

The solution of (31) consists of the functions \( u_n(t) = u_n^d(t) \) defined by

\[
u_n^d(t) = b^{-1}(t) \left\{ \left[ x_n^d(t) - x_n(t)\right]|_{T_n} + [\pi^2 n^2 - c(t)]x_n(t) - w_n(t) \right\}, \quad \forall \ n = 0, 1, \ldots
\]

(32)

where \( u_n^d(t) \) is called the inverse dynamics (id) solution. As a result, we see that the distributed control function

\[
u_n^d(z, t) = \sum_{n=0}^{\infty} u_n^d(t) \varphi_n(z)
\]

(33)

\[
= b^{-1}(t) \sum_{n=0}^{\infty} \left[ \left[ x_n^d(t) - x_n(t)\right]|_{T_n} + [\pi^2 n^2 - c(t)]x_n(t) - w_n(t) \right] \varphi_n(z)
\]

gives the desired behavior of transients \( x(z, t) \to x_n^d(z, t) \).

Assume that the series (33) is absolutely convergent and a certain value \( M_n \) exists such that the requirement (14) is satisfied, that is the realizability condition of the desired behavior in the discussed PDE system.

B. Control law and closed-loop system

In order to ensure that \( e_n^F = 0 \) when the parameters \( c(t), b(t) \) are varying and unknown as well as the external distributed disturbance \( w(z, t) \) is unavailable for measurement, let us consider the control law for the equation of the time function \( x_n(t) \) with \( x_n^d(t) \) in feedback, that is

\[
\mu_n u_n(t) + d_n u_n = k_n \left\{ \left[ x_n^d(t) - x_n(t)\right]|_{T_n} \right\}, \quad \forall \ n = 0, 1, \ldots
\]

(34)

where \( u_n(0) = u_n^0 \) and \( \mu_n > 0, \ d_n > 0, \ d_n > 0 \). Note that the control law given by (34) corresponds to a proportional-integral (PI) controller. Hence, the closed-loop system equations for the \( n \)th mode are given by

\[
\dot{x}_n(t) = [c - \pi^2 n^2 + w_n + b u_n],
\]

\[
\mu_n u_n(t) + d_n u_n = k_n \left\{ \left[ x_n^d(t) - x_n(t)\right]|_{T_n} \right\}, \quad \forall \ n = 0, 1, \ldots
\]

(36)

Substitution of (35) into (36) yields the closed-loop system equations in the form

\[
x_n^d(t) = [c - \pi^2 n^2 + w_n + b u_n],
\]

\[
\mu_n u_n(t) + d_n u_n = k_n \left\{ \left[ x_n^d(t) - x_n(t)\right]|_{T_n} - [c - \pi^2 n^2]x_n(t) - w_n(t) \right\}, \quad \forall \ n = 0, 1, \ldots
\]

(38)

Since \( \mu_n \) is a small parameter, the closed-loop system equations (37)–(38) are the singularly perturbed equations. If \( \mu_n \to 0 \), then fast and slow modes appear in the closed-loop system (37)–(38) and the time-scale separation between these modes depends on \( \mu_n \).

C. Two-time-scale motions analysis

By applying the method of time-scale separation [22], from (37)–(38), we get the equation of the fast-motion subsystem (FMS) for the \( n \)th mode, that is

\[
u_n u_n(t) + d_n u_n + k_n b u_n = k_n \left\{ \left[ x_n^d - x_n(t)\right]|_{T_n} \right\}, \quad \forall \ n = 0, 1, \ldots
\]

(39)

where \( x_n(0) = x_n^0, \ u_n(0) = u_n^0 \), \( \forall \ n = 0, 1, \ldots \) Substitution of (39) into (38) yields the closed-loop system equations in the form

\[
x_n^d(t) = [c - \pi^2 n^2 + w_n + b u_n],
\]

\[
\mu_n u_n(t) + d_n u_n = k_n \left\{ \left[ x_n^d - x_n(t)\right]|_{T_n} - [c - \pi^2 n^2]x_n(t) - w_n(t) \right\}, \quad \forall \ n = 0, 1, \ldots
\]

(38)

Assume that the asymptotic stability of the FMS unique equilibrium point holds and desired sufficiently small settling time of the transients of \( u_n(t) \) can be achieved by selection of \( \mu_n \).

Let us obtain an equation of the slow-motion subsystem (SMS) under the condition of FMS stability. After the rapid decay of transients in (39), we have the steady state (more precisely, quasi-steady state) for the FMS (39). In particular, if \( \mu_n \to 0 \) in (39) with \( d_n = 0 \), then we obtain

\[
u_n(t) = u_n^d(t)
\]

where \( u_n^d(t) \) is given by (32). Substitution of (32) into the right member of (37) yields the SMS which is the same as the desired differential equation given by (29). Hence, the desired behavior for \( x_n(t) \) is fulfilled after fast
damping of the FMS transients despite the mentioned above uncertainties. Moreover, if $d_{n,0} = 0$, then the integral action is incorporated into the control loop and, accordingly, the robust zero steady-state error is maintained.

V. SAMPLED-DATA CONTROL SYSTEM WITH TTSM

A. Control problem

In this section the discrete-time counterpart of the above design methodology is discussed.

Let us consider the backward approximation of (4) preceded by a zero-order hold (ZOH) with the sampling period $T_s$, that is

$$x_n[k] = x_n[k-1] + T_s \{ f_n[k-1] + b[k-1]u_n[k-1] \}, \quad (40)$$

where $k = 0, 1, 2, \ldots$. Here we have

$$f_n[k-1] = \{ c[k-1] - x_n^2[k-1] \} x_n[k-1] + w_n[k-1]$$

and $x_n[k], w_n[k], u_n[k], c[k], b[k]$ represent samples of $x_n(t)$, $w_n(t), u_n(t), c(t), b(t)$ at $t = kT_s$, respectively.

The objective is to design a control system having

$$\lim_{k \to +\infty} e_n[k] = 0. \quad (41)$$

Here $e_n[k] = r_n[k] - x_n[k]$ is the error of the reference input realization, $r_n[k]$ being the samples of the reference input $r(t)$, where the control transients $e_n[k] \to 0$ should meet the desired performance specifications given by (29).

By a Z-transform of (29) preceded by a ZOH, the desired pulse transfer function

$$H_n^d(z) = \frac{z - 1}{z} \left\{ e^{-\frac{T_n}{s+1/T_n}} \right\} \bigg|_{s=kT_s}$$

follows. Hence, from (42), the desired stable difference equation

$$x_n[k] = x_n[k-1] + T_s a_n(T_s) \{ r_n[k-1] - x_n[k-1] \} \quad (43)$$

results, where

$$a_n(T_s) = \frac{1 - e^{-T_s/T_n}}{z - e^{-T_s/T_n}} \quad (44)$$

and the response of (43) corresponds to the assigned transient performance indices.

Let us rewrite, for short, the desired difference equation (43) as

$$x_n[k] = F(x_n[k-1], r_n[k-1]), \quad (45)$$

where we have $x_n[k] = r_n[k]$ at the equilibrium of (45) for $r_n[k] = \text{const}$, $\forall k$. Denote

$$e_n^F[k] = F(x_n[k-1], r_n[k-1]) - x_n[k], \quad (46)$$

where $e_n^F[k]$ is the realization error of the desired dynamics assigned by (45). Accordingly, if for all $k = 0, 1, \ldots$ the condition

$$e_n^F[k] = 0 \quad (47)$$

holds, then the desired behavior of $x_n[k]$ with the prescribed dynamics of (45) is fulfilled. The expression (47) is the insensitivity condition for the $x_n[k]$ transient performance with respect to the external disturbance and varying parameters of the plant model given by (40). In other words, the control design problem (41) has been reformulated as the requirement (47). The insensitivity condition given by (47) is the discrete-time counterpart of (30).

B. Discrete-time control law

Let us consider the following control law:

$$u_n[k] = u_n[k-1] + \lambda_n(T_s) \{ F(x_n[k-1], r_n[k-1]) - x_n[k-1] \}, \quad (48)$$

where $\lambda_n(T_s) = T_s^{-1} \tilde{\lambda}_n$ and the reference model of the desired output behavior is given by (43). In accordance with (43) and (46), the control law (48) can be rewritten as the difference equation

$$u_n[k] = u_n[k-1] + \tilde{\lambda}_n \{ a_n(T_s) \{ r_n[k-1] - x_n[k-1] \} - x_n[k-1] \} - x_n[k-1] - [x_n[k] - x_n[k-1]]/T_s \quad (49)$$

The control law (49) is the discrete-time counterpart of the continuous-time control law (34).

C. Two-time-scale motions analysis

The closed-loop system equations for the $n$th mode have the form:

$$x_n[k] = x_n[k-1] + T_s \{ f_n[k-1] + b[k-1]u_n[k-1] \}, \quad (50)$$

$$u_n[k] = u_n[k-1] + \tilde{\lambda}_n \{ a_n(T_s) \{ r_n[k-1] - x_n[k-1] \} - x_n[k-1] - [x_n[k] - x_n[k-1]]/T_s \quad (51)$$

First, note that the stability and the rate of the transients of $u_n[k]$ in (52)–(53) depend on the controller parameter $\tilde{\lambda}_n$. Second, $x_n[k] - x_n[k-1] \to 0$, $f_n[k] - f_n[k-1] \to 0$, and $b[k] - b[k-1] \to 0$ as $T_s \to 0$. Hence, we have a slow rate of the transients of $x_n[k]$ as $T_s \to 0$. Thus, if $T_s$ is sufficiently small, the two-time-scale transients are induced in the closed-loop system (52)–(53), where the FMS for the $n$th mode is governed by

$$u_n[k] = \{ 1 - \tilde{\lambda}_n \} \{ u_n[k-1] + \tilde{\lambda}_n \{ a_n(T_s) \{ r_n[k-1] - x_n[k-1] \} - x_n[k-1] \} - f_n[k-1] \} \quad (54)$$

and $x_n[k] = x_n[k-1]$, $f_n[k] = f_n[k-1]$, i.e., $x_n[k] = \text{const}$, $f_n[k] = \text{const}$, as well as $a = b[k] = \text{const}$ (frozen variables) during the transients in the FMS (54).

From (54), the FMS characteristic polynomial

$$z - 1 - \tilde{\lambda}_n b \quad (55)$$

results, where its root lies inside the unit disk (hence, the FMS is stable) if $0 < \tilde{\lambda}_n < 2/b$. To ensure stability and fastest transient processes of $u_n[k]$, let us take the controller
parameter $\lambda_n = 1/h$, then the root of (55) is placed at the origin. Hence, the deadbeat response of the FMS (54) for the $n$th mode is provided.

Third, assume that the FMS (54) is stable and consider its steady state (quasi-steady state), i.e.,

$$u_n[k] - u_n[k-1] = 0.$$  

Then, from (54) and (56), we get

$$u_n[k] = u_n^*$$.  

Then, from (54) and (56), we get

$$u_n^*[k] = b^{-1}[k\left\{ a_n(T_s)[r_n[k-1] - x_n[k-1]] - f_n[k-1]\right\}].$$  

Substitution of (56) and (57) into (52) yields the SMS of (52)–(53), which is the same as the desired difference equation (43). Thus, if the degree of time-scale separation between fast and slow motions in the closed-loop system (52)–(53) is sufficiently large and the FMS transients are stable, then after the fast transients have vanished, the behavior of $x_n[k]$ tends to the solution of the reference model (43). Accordingly, the controlled transient process meets the desired performance specifications.

VI. CONCLUSION

The control problem for systems governed by one-dimensional parabolic equation has been discussed in the paper, while the presented results can be extended for other types of PDE systems. For instance, the presented approach to control system design can be directly applied to stabilization problem of the reaction-convection-diffusion system with exothermic reactions of generic kinetics discussed in [19, 20], where the mathematical model for the set of the first controllable modes has the same structure as (4). The main advantage of the presented approach is that the desired transient performance indices and control accuracy for the controlled modes are guaranteed despite unknown external disturbance and varying parameters of the PDE system. Hence, the presented design methodology can be useful for real-time control system design under uncertainties.

REFERENCES


