Decentralised two-time-scale motions control based on generalised sampling

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Abstract: A design method for the decentralised time-varying discrete-time output-feedback control of linear time-invariant plants with unstable unstructured decentralised fixed modes (UDFM) is introduced. The design method uses generalised sampled-data hold functions to eliminate the UDFMs and to decouple the discrete-time equivalent model of the plant into independent input–output channels. Through this structural change, the plant becomes suitable for a stabilising high-sampling-rate controller that induces two-time-scale motions (TTSM) in the closed-loop system. As a result, the discrete-time controller is likewise decoupled into distinct local agents and the TTSM closed-loop system is decentralised.

1 Introduction

Most of the existing results on the output feedback control of decentralised linear time-invariant (LTI) systems focus on the properties of the system such as stabilisability by means of LTI or linear time-varying (LTV) controllers [1, 2] and design techniques based on such properties. All of these existing methods consider the interconnected subsystems as one big system and then propose the design of local controllers for each subsystem by taking the effect of interconnections into account. Consequently, most of these methods involve complex algorithms and long iterative procedures [3, 4]. Even of more importance, some of them have limited applications because they cannot overcome the limitations imposed by the structural properties of the plant [2].

The most significant obstacle in the design of decentralised output-feedback controllers for certain class of LTI plants is the presence of unstable decentralised fixed mode (DFM), which LTI controllers cannot stabilise [2, 5]. DFMs can be classified as being either ‘structured’ or ‘unstructured’ [2, 5]. Structured DFMs are the modes that remain ‘fixed’ under any type of decentralised output-feedback control, including nonlinear and time-varying control. Therefore systems with unstable structured DFMs cannot be stabilised by any type of nonlinear or time-varying decentralised controller. Unstructured DFMs (UDFM), on the other hand, are the modes that can be eliminated through appropriate time-varying control laws. Since the combination of an ideal sampler, a discrete-time LTI controller and a hold operator acts as an LTV controller for the original continuous-time system, sampling can remove the non-zero and distinct UDFMs for almost all sampling rates [5].

Besides removing UDFMs, sampling can also modify the structure of the digraph of a plant. This property of sampling can be used to simplify the decentralised control design. For example, one can use generalised sampled-data hold functions (GSHF) to change the structure of the system to a hierarchical form [6]. The equivalent discrete-time system can then be stabilised by a series of smaller decentralised controllers that one can design by using centralised control design methods for each subsystem, independently. An approach is proposed in [7] to design a near-optimal GSHF for decentralised control systems. The approach uses a finite set of basis functions for constructing the desired hold function. This method is further developed in [8] to design a high-performance decentralised simultaneous stabiliser for a finite set of continuous-time systems using the linear matrix inequality (LMI) technique.

The robustness of control via GSHF has been previously studied [9–11]. Of particular interest are the results in [12], which address the robustness of zero-shifting via GSHFs. These results are of special interest because DFM is, in fact, transmission zeros of the system and a set of its subsystems [3]. Therefore the conditions stated in [12] directly apply to the decentralised control problem studied here.

One centralised control design method that can be effectively combined with GSHF-based structural modification to achieve decentralised output-feedback control is the two-time-scale motions (TTSM) control method [13]. The main advantages of this method are its robustness to parameter variations and its disturbance rejection capability [13–15]. Elaborating on the results introduced in [16], the present development incorporates generalised sampling into the original TTSM control method to provide a simple solution to the decentralised output-feedback stabilisation problem for multi-input multi-output (MIMO) linear plants. Such a solution encompasses the design of GSHFs for the structural modification of the system, as well as the design of a controller whose dynamics are much faster than those of the plant. The structural modification has two purposes:

(i) to eliminate unstable UDFMs from the discrete-time equivalent system so that it complies with the requirements of TTSM control; and
(ii) to decompose the discrete-time equivalent model into independent subsystems so that local stabilising controllers for each of them can be designed separately.

The main contribution of the present paper is the use of GSHFs for the design of the structure of discrete-time equivalent systems such that decentralised TTSM control can be implemented. The contribution capitalises on some of the important advantages of GSHFs, namely the elimination of UDFMs, performance improvement and arbitrary pole assignment, as reported in [17–19].

The remainder of this paper is organised as follows. In Section 2 the control objective is introduced. Then in Section 3 the problem is formulated and some definitions which are essential in the development of the main results are introduced. The proposed decentralised two-time-scale motion control strategy using sampled-data hold functions is presented in Section 4. Robustness of the proposed controller with respect to the parameter variation is briefly investigated in Section 5. The results obtained are validated by the simulations in Section 6. Finally in Section 7 some concluding remarks are drawn.

2 Control problem

Consider square plants of the form

\[ x(t) = Ax(t) + Bu(t) \]
\[ y(t) = Cx(t) \] (1)

in which \( x(t) = [x_1(t), \ldots, x_n(t)]^T \) is the state, and \( u(t) = [u_1(t), \ldots, u_m(t)]^T \) and \( y(t) = [y_1(t), \ldots, y_m(t)]^T \) are the input and output vectors, respectively.

**Assumption 1:** Let matrices \( B \) and \( C \) be rank \( m \). Unstable DFM’s may be present, provided they are unstructured, nonzero, and distinct (see [5]).

The control objective can be defined in terms of the sampled equivalent system as follows. Let \( T_s \) be the sampling period. Let also \( y[k] = y(t)|_{t=kT_s} \) represent the samples of the output of the system (1) and \( r[k] = r(t)|_{t=kT_s} \) represent the samples of the associated reference signal. The controller must then ensure that

\[ \lim_{k \to \infty} (r[k] - y[k]) = 0 \] (2)

is satisfied for constant values of \( r(t) \). Furthermore, assume that a decentralised output-feedback control of the form

\[ u_i(t) = \kappa_i(t, y[0, t], r_i[0, t]), \quad i \in \tilde{m} = \{1, \ldots, m\} \] (3)

is required even in the presence of unstable UDFMs.

**Notation:** Throughout this paper, scalars, matrices and vectors will be denoted by means of small, capital and bold-small symbols, respectively. Laplace transforms will also be represented by capital letters. Furthermore, the set of integers \([0, 1, 2, \ldots]\) will be denoted by \( \mathbb{Z} \) and the symbol \( \tilde{m} \) will be used to represent the set \([1, \ldots, m]\).

3 Sampled-data system

3.1 Plant restructuring and normal form

A GSHF \( \phi(t) \) is a periodic function of time with period \( T_s \) that multiplies the output of a zero-order hold (ZOH). In other words, the result of applying the GSHF \( \phi(t) \) to the signal \( \tilde{u}(t) \) is given by \( u(t) = \phi(t) \tilde{u}(t) \), where \( \tilde{u}(t), k \in \mathbb{Z}, \) maintains the constant value \( \tilde{u}[k], k = 0, 1, \ldots \), the sample of \( \tilde{u}(t) \) at \( t = kT_s \) during the sampling interval \( \tau = [kT_s, (k + 1)T_s] \). An example of the sampling of an arbitrary continuous-time signal \( \tilde{w}(t) \) with a sinusoidal GSHF is shown in Fig. 1.

Let \( (C, A_d, B_d) \) denote the discrete-time equivalent model of the system \((C, A, B)\) given by (1), when the input signal \( u(t) \) is generated by the \( m \) GSHFs

\[ \Phi(t) = \text{diag}([\phi_1(t), \ldots, \phi_m(t)]) \]

and the samples \( \tilde{u}[k] \), such that

\[ u(t) = \Phi(t) \tilde{u}[k] \] (4)

The discretised equations of the system (1) are then given by

\[ x[k + 1] = A_dx[k] + B_d\tilde{u}[k] \]
\[ y[k] = Cx[k] \] (5)

where \( A_d \) and \( B_d \) are given by

\[ A_d = e^{AT_s} \]
\[ B_d = \int_0^{T_s} e^{(AT_s-t)B} \Phi(t) \, dt \] (6a)

These equations suggest that the functions \( \phi_i(t), i \in \tilde{m}, \) are parameters in the design of the matrix \( B_d \). In fact, the only requirement for such design is that each of the \( m \) columns

\[ b_{di} = \int_0^{T_s} e^{(AT_s-t)B_i} \phi_i(t) \, dt, \quad i \in \tilde{m} \] (7)

of \( B_d = [b_{d1}, \ldots, b_{dm}] \) must belong to the controllable subspace of the respective subsystem \((A_d, b_i), i \in \tilde{m} [17, 20] \).

Subject to this constraint, we suppose the following.

**Assumption 2:** Without loss of generality, let system (1) be given in the Kalman observable canonical form. Then \( C = [C_0 \ 0] \) and rank\((C_0) = m \). Define the matrix \( B_d = [B_{d1}^T \ B_{d2}^T]^T \). Assume that GSHFs \( \phi_i(t) \) for each input channel \( u_i(t), i \in \tilde{m}, \) exist such that rank\((B_{d1}) = m \). Then the matrix \( CB_d = C_0B_{d1} \) is invertible.

In order to ensure that rank\((B_{d1}) = m \), one must first compute basis for the controllable spaces \( C_i \) of the subsystems \((A, b_i), i \in \tilde{m} \). One must verify whether \( m \) vectors \( v_i, i \in \tilde{m}, \) exist such that \( v_i \in C_i \) and such that \( B_d = [v_1, \ldots, v_m] \) satisfies rank\((B_{d1}) = m \). If these vectors exist, then one can find the \( m \) GSHFs \( \phi_i(t) \) that satisfy (7) for \( b_{di} = v_i, i \in \tilde{m} \). A detailed algorithm to find GSHFs
such the matrix $C_B$ is diagonal and invertible is presented in Section 4.4.

Under Assumption 2, consider an invertible similarity transformation given by the matrix

$$W = \begin{bmatrix} C \\ \tilde{C} \end{bmatrix}$$

(8)

with $\tilde{C}$ chosen such that $W$ is full rank. If this transformation is applied to (5), the discrete-time equivalent system (DTES) takes the normal form

$$y[k+1] = A_{11}y[k] + A_{12}z[k] + B_1\tilde{u}[k]$$

(9a)

$$z[k+1] = A_{12}y[k] + A_{22}z[k] + B_2\tilde{u}[k]$$

(9b)

where $[y^T, z^T]^T = Wx, z \in \mathbb{R}^{n-m}, B_1 = CB_d$. Equation (9) will be used to separate the external and internal dynamics of the plant, as described in [15].

### 3.2 Desired model and desired output

Assume that, for each of the input–output channels, the desired model is given by the following stable transfer function

$$H_i^*(s) = \frac{\theta_i}{s + \theta_i}, \quad i \in \tilde{m}$$

(10)

where $\theta_i > 0, i \in \tilde{m}$, are design parameters that determine the time constant of the desired model for the $i$th input–output agent. From (10), the $i$th desired pulse transfer function is

$$\hat{H}_i^*(z) = \frac{z-1}{z} z \left\{ \frac{\theta_i}{s(s+\theta_i)} \right\}$$

$$\hat{H}_i^*(z) = \frac{1-e^{-\theta_iT\tau}}{z - e^{-\theta_iT\tau}}, \quad i \in \tilde{m}$$

If $\hat{H}_i^*(z)$ given above governs the actual output, then $y_i[k+1]$ is given by

$$y_i[k+1] = e^{-\theta_iT\tau}y_i[k] + r_i[k](1-e^{-\theta_iT\tau})$$

$$= y_i[k] + (1-e^{-\theta_iT\tau})(r_i[k] + y_i[k])$$

$$\triangleq F_i(y_i[k], r_i[k]), \quad i \in \tilde{m}$$

(11)

where the condition $r_i[k] = y_i[k]$ holds at the equilibrium of (11) for a constant $r_i[k], \forall k \in \mathbb{Z}$.

Let us rewrite (11) in the vector form given by

$$y[k+1] = y[k] + (I_m - e^{-\Theta_T\tau})(r[k] - y[k])$$

$$\triangleq F(y[k], r[k])$$

(12)

where $\Theta = \text{diag} \{ \theta_1, \ldots, \theta_m \}$ and $y[k] = [y_1[k], \ldots, y_m[k]]^T$, and $r[k] = [r_1[k], \ldots, r_m[k]]^T$. Denote

$$e^F[k] = F(y[k-1], r[k-1]) - y[k]$$

where $e^F[k]$ is the realisation error of the desired behaviour assigned by (12). Accordingly, if for all $k \in \mathbb{Z}$ the condition

$$e^F[k] = 0$$

(13)

holds, then the desired output behaviour of $y[k]$ is fulfilled. Hence, the control problem of output regulation given by (2) has been reformulated as the requirement (13).

### 3.3 Internal dynamics

Under Assumption 2, it follows from (9a) and (12) that the solution to (13) is given by

$$u_i[k] = B_{i-1}^{-1}(F_i(y[k], r[k]) - A_{11}y[k] - A_{12}z[k])$$

(14)

and is called the solution of the inverse dynamics. Note that this solution corresponds to the reference input-controlled output map assigned by (12). Let $u[k] = u_i[k]$ and substitute (14) into (9) to obtain

$$y[k+1] = F(y[k], r[k])$$

(15a)

$$z[k+1] = (A_{22} - B_2B_1^{-1}A_{12})z[k]$$

$$+ (A_{21} - B_2B_1^{-1}A_{11})y[k]$$

$$+ B_2B_{i-1}^{-1}F_i(y[k], r[k])$$

(15b)

These equations describe the behaviour of the state variables $y[k], z[k]$ under the condition that the desired output behaviour assigned by (12) occurs. From (15), one can derive the zero-dynamics equations by taking $y[k+1] = y[k] = r[k] = 0$. In system (15), the external subsystem (15a) equals the reference model equation (12). Equation (15b) is the internal subsystem equation, whose characteristic polynomial is given by

$$\det \{ \mathcal{I}_{n-m} - A_{22} + B_2B_{i-1}^{-1}A_{12} \}$$

(16)

System (9) is said to be internally stable if and only if all roots of the internal subsystem characteristic polynomial (16) lie in the unit disk. The stability of the internal subsystem (15b) depends only on the inherent properties of the system (9).

**Assumption 3:** Let the discrete-time equivalent system (9) be such that all the roots of the internal subsystem characteristic polynomial (16) lie inside the open unit disk.

Remark 1: Under Assumption 3, it follows from (12) that the zero-dynamics equations (15) imply the zero input-controlled output map assigned by (12), and thus, all the roots of the internal subsystem characteristic polynomials (16) lie inside the open unit disk.

4 Main result

4.1 Control law

Define

$$G \triangleq B_1 = CB_d$$

Recalling Assumption 2, consider (4) and the feedback transformation

$$\tilde{u}[k] = T_iG^{-1}v[k]$$

(17)

where $v[k]^T = [v_1[k], \ldots, v_m[k]]^T$. Let the control law be given by

$$v_i[k+1] = v_i[k] + \lambda_i(F_i(y_i[k+1], r_i[k+1]) - y_i[k+1]), \quad i \in \tilde{m}$$

(18)

where $\lambda_i, i \in \tilde{m}$ are non-zero constants. Define also non-zero constants $\bar{\lambda}_i, i \in \tilde{m}$, such that $\lambda_i = T_i^{-1}\bar{\lambda}_i$. The design of the parameters $\lambda_i$ is discussed below (see Section 4.3). In agreement with (11), rewrite (18) in the
vector form given by
\[ v[k + 1] = v[k] + T_s^{-1} \lambda [y[k] - y[k + 1]] 
+ (I_m - e^{-\Theta T_s})(r[k] - y[k]) \] (19)
where \( \lambda = \text{diag} \{ \lambda_1, \ldots, \lambda_m \} \), \( \Theta = \text{diag} \{ \theta_1, \ldots, \theta_m \} \). Note that, first, (19) is a causal controller implemented without prediction, and second, the condition (13) holds at the equilibrium of (19) when \( v[k + 1] = v[k] \).

4.2 Closed-loop system

It follows from (5) and (6) that the output of the system (9) can be rewritten as
\[ y[k + 1] = CA_x \dot{x}[k] + CB_{\tilde{u}} \tilde{u}[k] \]
\[ = y[k] + T_s (C A x[k] + C B_{\tilde{u}} \tilde{u}[k]) \] (20)
where
\[ M = \sum_{j=1}^{\infty} \frac{T_s}{j!} A^j, \quad B_{\tilde{d}} = T_s^{-1} B \]
and \( T_s \) plays the role of a small parameter.

Then the transformation (17) applied to (20) yields
\[ y[k + 1] = y[k] + T_s (C A x[k] + v[k]) \] (21)
Thus, (21) can be decoupled with respect to the inputs into individual components as
\[ y_i[k + 1] = y_i[k] + T_s (c_i A x[k] + v_i[k]), \quad i \in \tilde{m} \] (22)
From (21), the equation
\[ y[k + 1] = y[k] + T_s (C A x[k] + C M z[k] + v[k]) \] (23)
results, where \( M_1 \in \mathbb{R}^{n \times m} \) and \( M_2 \in \mathbb{R}^{n \times (n - m)} \) are such that \( MW^{-1} = [M_1, M_2] \). Note that the comparison of (9a) with (23) yields
\[ A_{11} = I_m + T_s C M_1, \quad A_{12} = T_s C M_2 \] (24)
Equation (23) implies that \( y[k + 1] - y[k] \rightarrow 0 \) as \( T_s \rightarrow 0 \). Hence, for small sampling periods, the output trajectory varies slowly. This follows from (6a), which shows that \( A_0 = I_m + T_s M' \rightarrow I_m \) as \( T_s \rightarrow 0 \). Throughout this paper, it will be assumed that \( T_s \) satisfies this condition by being chosen according to the following definition.

Definition 1: A sampling time \( T_s \) is deemed small if \( T_s \ll \tau_{\text{min}} \), where \( \tau_{\text{min}} \) is the time constant associated with the largest pole of the system (1). In practice \( T_s \leq 0.1 \tau_{\text{min}} \) suffices typically.

By taking into account (9a), (9b), (17), (23), and (19), one obtains the closed-loop system equations as
\[ y[k + 1] = y[k] + T_s (C A x[k] + C M_2 z[k] + v[k]) \] (25a)
\[ z[k + 1] = A_{21} y[k] + A_{22} z[k] + T_s B_{12} G^{-1} v[k] \] (25b)
\[ v[k + 1] = v[k] + T_s^{-1} \lambda [y[k] - y[k + 1]] 
+ (I_m - e^{-\Theta T_s})(r[k] - y[k]) \] (25c)

4.3 TSM analysis

Next, in order to analyse the closed-loop system properties, let us replace \( y[k + 1] \) in (25c) by the right side of (25a). This substitution yields the following equations
\[ y[k + 1] = y[k] + T_s (C A x[k] + C M_2 z[k] + v[k]) \] (26a)
\[ z[k + 1] = A_{21} y[k] + A_{22} z[k] + T_s B_{12} G^{-1} v[k] \] (26b)
\[ v[k + 1] = (I_m - \tilde{\lambda} w[k] - \tilde{\lambda} C M_2 z[k] 
+ T_s^{-1} \tilde{\lambda} (I_m - e^{-\Theta T_s})(r[k] - y[k]) \] (26c)

Notice that, from (6a), \( A_0 \rightarrow I_m \) as \( T_s \rightarrow 0 \). It then follows from (5) that \( x[k + 1] - x[k] \rightarrow 0 \) as \( T_s \rightarrow 0 \). Hence, \( y[k + 1] - y[k] \rightarrow 0 \) and \( z[k + 1] - z[k] \rightarrow 0 \) as \( T_s \rightarrow 0 \).

On the other hand, the stability of transients for \( v[k] \) does not depend on \( T_s \) but on the design parameters \( \lambda_i \) of the matrix \( \Lambda \). Moreover, the rate of transients for \( v[k] \) increases as \( T_s \rightarrow 0 \). This implies that, for sufficiently small values of \( T_s \), the closed-loop system given by (26) consists of a slow-motion subsystem (SMS) with states \( y \) and \( z \), and a fast-motion subsystem (FMS) with state \( v \). From (26), the FMS in given by
\[ v[k + 1] = (I_m - \tilde{\lambda} w[k] - \tilde{\lambda} C M_2 z[k] 
+ T_s^{-1} \tilde{\lambda} (I_m - e^{-\Theta T_s})(r[k] - y[k]) \] (27)
where \( z[k], r[k] \) and \( y[k] \) are treated as constant values during the transients in (27). This requirement can be easily satisfied for sufficiently small sampling periods \( T_s \), as discussed in Definition 1 and in previous works on two-time scale analysis [13, 21]. Note that \( T_s^{-1} (I_m - e^{-\Theta T_s}) \rightarrow \Theta \) as \( T_s \rightarrow 0 \).

From (27), the characteristic polynomial of the FMS reads as
\[ P(z) = P_1(z) P_2(z) \cdots P_m(z) \]

![Fig. 2](image-url) Closed-loop system with feedback applied through ZOHs and the GSHFs \( \phi_{i}(t), \ldots, \phi_{m}(t) \) at the inputs \( u_i(t), \ldots, u_m(t) \)
where
\[ P_i(z) = z - 1 + \hat{\lambda}_i, \quad i \in \hat{m} \]

In order to stabilise (27), let the parameters \( \hat{\lambda}_i, i \in \hat{m} \), be such that these polynomials are stable. In particular, set the roots of the polynomials \( P_i(z) \) at the origin by taking \( \hat{\lambda}_i = 1, i \in \hat{m} \), which leads to the deadbeat response of the FMS (27).

From (27), the quasi-steady state is given by
\[ v_i^{eq}[t] = -CM_i\dot{y}[k] - CM_2z[k] + T_s^{-1}(I_m - e^{-\theta r})r[k] - y[k] \tag{28} \]
where \( v_i[k + 1] = v_i[k] = v_i^{eq} \). Hence, by substituting (28) into (26) and using (24), one can obtain the SMS given by (15).

**Theorem 1:** Let Assumption 1, 2, and 3 hold. If: (i) \( \theta_i > 0, i \in \hat{m} \); (ii) the parameters \( \lambda_i, i \in \hat{m} \), are such that the matrix \( I_m - \hat{\Lambda} \) is Schur stable; and (iii) \( T_s \) is as described in Definition 1, then the equilibrium point of the closed-loop system (26) is exponentially stable for any constant reference signal \( r(t) \).

**Proof:** By means of the similarity transformation given by (8), one can rewrite the equations of the closed-loop system (26) as
\[
\begin{bmatrix}
\dot{x}[k + 1] \\
\dot{z}[k + 1] \\
v[k + 1]
\end{bmatrix} = \begin{bmatrix}
I_n + T_sA_{11} & T_sA_{12} \\
\hat{\lambda}_{21} & \hat{\lambda}_{22}
\end{bmatrix} \begin{bmatrix}
\dot{x}[k] \\
\dot{z}[k] \\
v[k]
\end{bmatrix} + \begin{bmatrix}
0 \\
\Lambda (I_m - e^{-\theta r})
\end{bmatrix} r[k]
\]

where
\[
\hat{\lambda}_{11} = \frac{1}{T_s} \begin{bmatrix}
T_sCMW^{-1} & \hat{A}_{12} \\
A_{21} & A_{22} - I_{n-m}
\end{bmatrix}, \quad \hat{\lambda}_{22} = \begin{bmatrix}
I_m \\
B_2B_1^{-1}
\end{bmatrix}
\]
\[
\hat{\lambda}_{21} = -\Lambda (CA_d - e^{-\theta r}C)W^{-1} \quad \hat{\lambda}_{22} = I_m - \hat{\Lambda}
\tag{29}
\]
\( \Lambda = \text{diag} [\lambda_1, \ldots, \lambda_n], \hat{\lambda} = T_s\hat{\lambda}, \) and \( \Theta = \text{diag} [\theta_1, \ldots, \theta_n] \).

Then, if \( T_s \) is sufficiently small, one can use the decoupling similarity transformation described in (22) in order to separate the slow and fast motions of the unforced system. The transformation is given by
\[
Q = \begin{bmatrix}
I_{n-m} - TL \\
T_NL
\end{bmatrix}
\]
where \( L \in \mathbb{R}^{(n-m)\times(n-m)} \) and \( N \in \mathbb{R}^{n\times(n-m)} \) are the solutions to
\[
0 = \hat{\lambda}_{22} + N - A_{22}L + T_s[\hat{A}_{11} - \hat{A}_{12}L]
\]
\[
0 = \hat{\lambda} \quad N - N[\hat{A}_{22} + T_s[\hat{A}_{11} - \hat{A}_{12}L]]N - T_sNL\hat{A}_{22}
\]

The transformation \( Q \in \mathbb{R}^{n\times n} \) is guaranteed to exist and allows one to write
\[
\begin{bmatrix}
\dot{x}[k + 1] \\
\dot{v}[k + 1]
\end{bmatrix} = \begin{bmatrix}
I_n + T_sA_d & 0 \\
0 & I_m - \hat{\Lambda}
\end{bmatrix} \begin{bmatrix}
\dot{x}[k] \\
\dot{v}[k]
\end{bmatrix}
\tag{30}
\]
where \( [\dot{x}, \dot{v}]^T = Q[x^T W^T, v^T]^T \) and
\[
A_d = \hat{A}_{11} + \hat{A}_{12}(I_m - A_{22}^{-1}\hat{A}_{21})
\tag{31}
\]

On one hand, the FMS is stable because the matrix \( I_m - \hat{\Lambda} \) is Schur stable by design. On the other hand, the modes of the slow-motion dynamics are, according to (30), given by the eigenvalues of the matrix \( I_n + T_sA_d \). By substituting (29) into (31), one obtains
\[
\begin{align*}
I_n + T_sA_d = I_n + T_s \begin{bmatrix}
A_{21} & A_{22} - I_{n-m}
\end{bmatrix} & + \begin{bmatrix}
I_m \\
B_2B_1^{-1}
\end{bmatrix} \hat{\lambda}_{22}^{-1}(-\Lambda (CA_d - e^{-\theta r}C)W^{-1}) \\
& = I_n + T_sCMW^{-1} & \\
& - \begin{bmatrix}
I_m \\
B_2B_1^{-1}
\end{bmatrix} (CA_d - e^{-\theta r}C)W^{-1}
\end{align*}
\]

Substitution of \( A_d = I_n + T_sM \) in the above equation and simplification of the resulting expression lead to
\[
\begin{align*}
I_n + T_sA_d = & \begin{bmatrix}
I_m \\
A_{21} & A_{22}
\end{bmatrix} & + \begin{bmatrix}
I_m \\
B_2B_1^{-1}
\end{bmatrix} & \begin{bmatrix}
0 \\
A_{21} & A_{22} - B_2B_1^{-1}e^{-\theta r}
\end{bmatrix} & \begin{bmatrix}
I_{n-m} - e^{-\theta r} & 0 \\
0 & I_m
\end{bmatrix}
\end{align*}
\]

which leads to
\[
I_n + T_sA_d = \begin{bmatrix}
A_{21} + B_2B_1^{-1}(e^{-\theta r} - A_{11}) & A_{22} - B_2B_1^{-1}A_{12}
\end{bmatrix}
\tag{32}
\]

The above equation implies that the modes of the SMS are given by the design parameters \( \theta_i, i \in \hat{m} \), and the eigenvalues of the matrix \( -B_2B_1^{-1}A_{12} \). These eigenvalues are in fact the modes of the internal dynamics. Therefore if Assumption 3 holds such that \( -B_2B_1^{-1}A_{12} \) is Schur stable, \( \theta_i > 0 \) for \( i \in \hat{m} \), \( T_s \rightarrow 0 \), and \( r(t) \) is constant for \( t > 0 \), then the equilibrium point of the closed-loop system (26) is exponentially stable. \( \square \)

### 4.4 Decentralisation

Consider again the system (1) and the controller (4), (17) and (19). It follows from (17), and from the fact that the \( i \)-th state of the control agent \( v_i \) given by (19) is related to the \( i \)-th output \( y_i \) only, that the closed-loop system can be decentralised if the matrix \( G \) is diagonal. To see this, substitute (17) into (4) to obtain
\[
u(t) = T_s\Phi(t)G^{-1}v[k]
\]

Since the matrix \( \Phi(t) \) is diagonal by definition, diagonality of \( G^{-1} \) guarantees that no coupling between different control agents exists. \( G \) can be made diagonal through the proper choice of \( \Phi(t) \) if the matrix \( C \) has certain properties. The next theorem provides necessary and sufficient
conditions for the existence of a decentralised TTSM controller (4), (17) and (19) for the system (1) or, equivalently, necessary and sufficient conditions for the existence of $\Phi(t)$ such that the matrix $G$ is diagonal.

**Theorem 2:** Let the conditions of Theorem 1 hold for the system (1). Denote by $C^\dagger \in \mathbb{R}^{m \times n}$ the right pseudo-inverse of $C$. Moreover, let $d_i$ be the $i$th column of $C^\dagger$ and $B_i$ be the $i$th column of $B$. If the controllable subspaces of $(A, b_i)$, denoted by $C_i$, are such that $d_i \in C_i$, $i \in \mathbb{N}$, then GSHFs exist such that the matrix $G$ is diagonal and hence a decentralised stabilising controller (4), (17), and (19) for the system (1) exists.

**Proof of necessity:** If the conditions of Theorem 1 hold, then a stabilising TTSM controller exists for the system (1). Moreover, the matrix $C$ has rank $m$ and the matrix $C^\dagger \in \mathbb{R}^{m \times n}$, the right pseudo-inverse of $C$, is such that $CC^\dagger = I_m$. If $d_i \in C_i$, then, by virtue of the results in [20], there exist GSHFs $\phi_i(t)$, $i \in \mathbb{N}$, such that

$$a_i d_i = \int_0^T e^{(T_e - \tau)B} \phi_i(\tau) \, d\tau$$

for some constants $a_i$. Consequently, the matrix $B_d$ can be designed as $B_d = [a_1 d_1, \ldots, a_m d_m]$, implying that $G = CB_d$

$$= C[d_1, \ldots, d_m] \text{diag}[a_1, \ldots, a_m]$$

$$= CC^\dagger \text{diag}[a_1, \ldots, a_m]$$

$$= \text{diag}[a_1, \ldots, a_m]$$

The transformation (17) then becomes

$$\bar{u}_i[k] = \frac{1}{a_i} v_i[k], \quad i \in \mathbb{N}$$

Since the state $v_i[k]$ depends only on $y_i$ and $r_i$, $i \in \mathbb{N}$, the controller is decoupled as

$$u_i(t) = \frac{\phi_i(t)}{a_i} v_i[k], \quad kT_s \leq t < (k + 1)T_s$$

for $i \in \mathbb{N}$, and the TTSM controller will conform to the decentralised structure specified by (3).

**Proof of necessity:** Evidently, if $d_i \not\in C_i$, $i \in \mathbb{N}$, then there is no matrix $B_d$ such that $G$ is diagonal and, hence, the control law cannot be decoupled as in (3). \hfill \Box

Let $C = U\Sigma V^T$, $\Sigma = \text{diag}[\sigma_1, \ldots, \sigma_m]$ represent the singular value decomposition of the matrix $C$ with $\sigma_1, \ldots, \sigma_m$ denoting its singular values. There exist $m$! different $\Sigma$ matrices for a given matrix $C$, and $U$ and $V$ are not unique. Since $C^\dagger = V \Sigma^T U^T$, there are at least $m$! different pseudo-inverses for $C$. One should then test if for these different $C^\dagger$ matrices the vectors $d_i$ belong to the controllable subspaces of $(A, b_i)$, for $i \in \mathbb{N}$.

The following algorithm can be used for the selection of GSHFs $\phi_i(t)$, $i \in \mathbb{N}$.

1. Find the singular values of $C^\dagger$ and assign them an arbitrary order.
2. Find the singular value decomposition of $C^\dagger$ for the given order of the singular values.
3. Verify if the columns $d_i$, $i \in \mathbb{N}$, of $C^\dagger$ belong to the controllable subspaces of $(A, b_i)$. If they do, go to step 5.
4. If not all possible orders of the singular values have been tested already, reorder the singular values and go to step 2. Else, stop, no solution exists.
5. Let $B_d = [a_1 d_1, \ldots, a_m d_m]$, where $a_i$ are design parameters.
6. Find $m$ GSHFs $\phi_i(t)$ so that (7) holds for $b_i = a_i d_i$, $i \in \mathbb{N}$, with $d_i$ as found in steps 2–4 (use, for instance, the methods presented in [17, 20]).

## 5 Robustness analysis

Assume that the system (1) is now subject to perturbations and disturbances such that

$$\dot{x}(t) = \hat{A}x(t) + \hat{B}u(t) + \hat{E}w(t)$$

$$\hat{y}(t) = \hat{C}x(t)$$

where $E \in \mathbb{R}^{m \times \eta}$, $\hat{y}(t)$ is the measured output, and $w(t) \in \mathbb{R}^\eta$ is a vector of unknown but bounded disturbances belonging to the set $\mathcal{W}_1$. The matrices $\hat{A} = A + \Delta_A$, $\hat{B} = B + \Delta_B$, and $\hat{C} = C + \Delta_C$ are constant perturbed counterparts of the nominal system matrices $A$, $B$, and $C$ with the perturbations $\Delta_A$, $\Delta_B$ and $\Delta_C$ satisfying the following assumption.

**Assumption 4:** The perturbations $\Delta_A$, $\Delta_B$, and $\Delta_C$ are bounded and belong to the known closed sets $\mathcal{A} \subset \mathbb{R}^{m \times \eta}$, $\mathcal{B} \subset \mathbb{R}^{n \times m}$, and $\mathcal{C} \subset \mathbb{R}^{m \times n}$, respectively.

The discrete-time equivalent perturbed system is represented by

$$x[k + 1] = \hat{A}_d x[k] + T_s \hat{B}_d \hat{u}[k] + w[k]$$

$$\hat{y}[k] = \hat{C}x[k]$$

with the discrete-time equivalent system matrices

$$\hat{A}_d = \hat{A}^T_s$$

$$T_s \hat{B}_d = \frac{1}{T_s} \int_0^{T_s} e^{(T_e - \tau)\hat{G}} \phi(t) \, dt$$

and the discrete-time equivalent disturbance

$$w[k] = \int_{kT_s}^{(k+1)T_s} e^{(k+1)T_e - \tau} \phi(t) \, dt$$

The control signal is now given by (4), (17) and

$$v[k + 1] = v[k] + T_s^{-1} \hat{A} \hat{e}^T[k + 1]$$

where $\hat{e}^T[k + 1] \triangleq \hat{F}(\hat{y}[k], r[k])$ is the realisation error of the desired behavior, and

$$\hat{F}(\hat{y}[k], r[k]) \triangleq \hat{y}[k] + (I_m - e^{-T_e}) (r[k] - \hat{y}[k])$$

is the desired $(k + 1)$th sample of the measured output.

The following theorem gives necessary and sufficient conditions for the existence of a TTSM controller for (35).

**Theorem 3:** Let the conditions in Theorem 1 hold for the system (1). Assume that $\phi_i(t)$, $i \in \mathbb{N}$, are finite $\forall t \in [0, T_s]$. Assume also that the sets $\hat{A}$, $\hat{B}$, and $\hat{C}$ are such that a diagonal matrix $\hat{A}$ exists with the property that (35) is internally stable and such that

$$A_F = I_m - \hat{A} \hat{C} \hat{B}_d (\hat{C} \hat{B}_d)^{-1}$$

4. If not all possible orders of the singular values have been tested already, reorder the singular values and go to step 2. Else, stop, no solution exists.
5. Let $B_d = [a_1 d_1, \ldots, a_m d_m]$, where $a_i$ are design parameters.
6. Find $m$ GSHFs $\phi_i(t)$ so that (7) holds for $b_i = a_i d_i$, $i \in \mathbb{N}$, with $d_i$ as found in steps 2–4 (use, for instance, the methods presented in [17, 20]).
is Schur stable for all $\Delta_i \in \tilde{A}, \Delta_B \in \tilde{B}$, and $\Delta_C \in \tilde{C}$. Then, a TIMS stabilizing controller for (35) exists such that, for the closed-loop system, $\tilde{y}[k] \rightarrow r[k] = \text{constant}$.

Proof: It is trivial to show that

$$\tilde{A}_d = e^{\tilde{A}T} = \sum_{j=0}^{\infty} \frac{T^j}{j!} \tilde{A}^j = I_n + T \tilde{M}$$

with

$$\tilde{M}_1, \tilde{M}_2 = \tilde{M} \tilde{W}^{-1} = \sum_{j=1}^{\infty} \frac{T^j}{j!} (A + \Delta_d)^j \tilde{W}^{-1}$$

and

$$\tilde{W} = \left[ \begin{array}{c} \tilde{C} \\ \tilde{C} \end{array} \right]$$

is an invertible similarity transformation. The algebraic manipulation of (35) allows us to rewrite the dynamics of the measured output as

$$\tilde{y}[k+1] = \tilde{y}[k] + T \tilde{C} \tilde{M}_1 \tilde{y}[k] + T \tilde{C} \tilde{M}_2 \tilde{y}[k] + T \tilde{C} \tilde{D}_d (C \tilde{B}_d)^{-1} \tilde{w}[k] + \tilde{C} \tilde{w}[k]$$

(41)

From the above relation and from substitution of (35) and (39) into (38) it follows that the fast-motion subsystem is governed by

$$v[k+1] = (A_m - \tilde{A} \tilde{B}_d (C \tilde{B}_d)^{-1}) v[k] + T_s^{-1} \tilde{A} (\tilde{C} \tilde{M}_1 \tilde{y}[k] + \tilde{C} \tilde{M}_2 \tilde{y}[k]) + T_s^{-1} \tilde{A} \tilde{C} \tilde{w}[k]$$

(42)

where $\tilde{z}[k], r[k], \tilde{y}[k]$ and $\tilde{y}[k]$ are treated as constant values during the transient time in (42).

Because of the perturbations, the dynamics of the FMS are not decoupled as in (27) where the matrix $I_m - \tilde{A}$ has diagonal form; instead, they are described by (42). The last equation shows that the roots of the characteristic polynomial of the FMS are the eigenvalues of the matrix $A_F$ given by (40). If there exist constants $\tilde{A}_i, i \in \tilde{m}$, such that this matrix is Schur stable for all $\Delta_i \in \tilde{A}, \Delta_B \in \tilde{B}$, and $\Delta_C \in \tilde{C}$, then the FMS is exponentially stable and reaches a quasi-steady state at which $v[k+1] = v[k] = v^{ss}$. Such quasi-steady state $v^{ss}$ is characterised by

$$v^{ss} = \frac{\tilde{C}}{T_s} \left( I_m - e^{-\Theta T_s} \right)^{-1} (r[k] - \tilde{y}[k]) + \tilde{C} \tilde{B}_d (C \tilde{B}_d)^{-1}$$

(43)

where, from (37) and the mean value theorem, $\sqrt{\left( \|w[k]/T_s\|_2 \right)}$ approaches a finite constant as $T_s \rightarrow 0$. Note that, from (36) and by virtue of Assumption 4, $\tilde{B}_d \in \tilde{B}_d \subset \mathbb{R}^{n \times n}$. By substituting (17), (43) and $v[k] = v^{ss}$ into (41), one can easily verify that, under this quasi-steady state, the slow-motion dynamics obey

$$\tilde{y}[k+1] = \tilde{y}[k] + \left( I_m - e^{-\Theta T_s} \right) (r[k] - \tilde{y}[k])$$

Thus, the dynamics of the measured tracking error $\tilde{g}[k] = r[k] - \tilde{y}[k]$ are described by

$$\tilde{g}[k+1] = e^{-\Theta T_s} \tilde{g}[k] + r[k+1] - r[k]$$

(44)

and the roots of the characteristic polynomial of the SMS are located at $z = e^{-\Theta T_s}$. The parameters $\theta_i, i \in \tilde{m}$, can then be used to arbitrarily place the modes of the closed-loop system, as long as they define a desired response that is sufficiently slow to allow for a time-scale separation. If $\theta_i$ is chosen as $\theta_i = T_i$, $i \in \tilde{m}$, then $T_i/T_s$ represents the degree of time-scale separation between the SMS and FMS, with $T = \min \left\{ T_i \right\}$. Then, if the reference signal is constant, such that $r[k+1] - r[k] = 0$, the internal stability of the perturbed system and (44) ensure that $\tilde{g}[k] = r[k] - \tilde{y}[k] \rightarrow 0$ and here $\tilde{y}[k]$ exponentially converges to $r[k]$. □

Another important fact to notice here is that the input matrix can be rewritten as

$$\tilde{B}_d = \tilde{B}_d + \Delta \tilde{B}_d$$

(45)

where $\tilde{B}_d$ is given by (6b) and

$$\Delta \tilde{B}_d = \frac{1}{T_s} \int_0^{T_s} e^{(r-T_s)\tilde{A}_d} \tilde{B}_d (\tilde{C} \tilde{B}_d)^{-1} \tilde{w} \Phi(t) dt + \frac{1}{T_s} \int_0^{T_s} \sum_{n=0}^{\infty} \frac{A^n - A^n}{n!} \tilde{B}_d (\tilde{C} \tilde{B}_d)^{-1} \tilde{w} \Phi(t) dt$$

(46)

This follows from the fact that $e^{(r-T_s)\tilde{A}_d}$ can be expressed as

$$e^{(r-T_s)\tilde{A}_d} = \sum_{n=0}^{\infty} \frac{A^n (T_s - t)^n}{n!}$$

and this in turn can be rewritten in the form

$$\sum_{n=0}^{\infty} \frac{A^n (T_s - t)^n}{n!} \triangleq \sum_{n=0}^{\infty} \frac{A^n (T_s - t)^n}{n!} + \Delta \tilde{A}_d$$

(47)

so that

$$e^{(r-T_s)\tilde{A}_d} = e^{(r-T_s)\tilde{A}_d} + \Delta \tilde{A}_d$$

(48)

Simple algebraic manipulation of (47) leads to

$$\Delta \tilde{A}_d = \sum_{n=0}^{\infty} \frac{A^n (T_s - t)^n}{n!} - \sum_{n=0}^{\infty} \frac{A^n (T_s - t)^n}{n!}$$

(49)

and substitution of (48) and (49) into (36) results in (46). Now, from (40) and the decomposition of $\tilde{B}_d$ in (45), it can be concluded that

$$A_F = I_m - \tilde{A} - \tilde{A} \tilde{C} \tilde{B}_d + \tilde{C} \tilde{D}_d (C \tilde{B}_d)^{-1}$$

(50)

Determining the existence of a matrix $\tilde{A}$ such that $A_F$ is Schur stable can be regarded as the robustness analysis of a perturbed system. This problem has been previously addressed (for e.g. see [23]) and therefore the details are omitted here.

In summary, if the matrices $\tilde{A}$ and $\Theta$ are chosen properly, the measured output of the closed-loop system exponentially tracks the desired constant reference signal and the actual tracking error depends on the measurement accuracy. The stability of the closed-loop system is insensitive to a class of bounded perturbations and time-varying disturbances as long as there is a sufficiently large time-scale separation between the rate of the transients of the FMS on the
The largest pole of the plant is located at only transmission zero of the system is located at controller is given by which is evidently decentralised.

\[ x(t) + \phi_1(t) = \theta_1 x(t) + \phi_2(t) \]

For the closed-loop system with non-zero initial conditions \( x(0) = [5, 5] \) and a step reference \( r[k] = [5, 5] \) applied at \( t = 5 \) s (i.e., \( r_1[k] = r_2[k] = u_s[k - 5] \)), where \( u_s[k] \) represents the unit step signal. The dashed lines in Figs. 3a and b represent the reference signal. These results show that the combined use of GSHFs and TTSM control can surmount the design limitations caused by the singularity of the ZOH matrix \( CB_d \).

The following example illustrates a decentralised design for a system that has an unstable UDFM.

Example 2: Assume that a decentralised controller must be designed for the plant (1) with matrices

\[
A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -9 & 9 & 0.95 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}
\]

and therefore the original TTSM design method cannot be applied. One can however use GSHFs to overcome this limitation. For example, let \( T_s = 0.1 \) s and consider the GSHFs

\[
\phi_1(t) = t^2 + 1200.5t - 59.02, \\
\phi_2(t) = t^2 - 0.1t - 0.99
\]

which solve (6b) such that

\[
B_d = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}
\]

and \( CB_d = \text{diag}[1, 1] \). The DTES is internally stable and the only transmission zero of the system is located at \( z = 0.90 \). The largest pole of the plant is located at \( s = 1 \); therefore the sampling period \( T_s = 0.1 \) s is sufficiently small according to Definition 1.

For \( \theta_1 = 1, \theta_2 = 1, \lambda_1 = 1 \) and \( \lambda_2 = 1 \), the TTSM controller is given by

\[
\begin{align*}
v_1[k + 1] &= v_1[k] + y_1[k] - y_1[k + 1] - 0.1(r_1[k] - v_1[k]) \\
v_2[k + 1] &= v_2[k] + y_2[k] - y_2[k + 1] - 0.1(r_2[k] - v_2[k]) \\
u_1(t) &= 0.1\phi_1(t)v_1[k] \\
u_2(t) &= 0.1\phi_2(t)v_2[k]
\end{align*}
\]

which is evidently decentralised.

Fig. 3 depicts the control signals and the output signals for the closed-loop system with non-zero initial conditions \( x(0) = [-5, 5, 5]^T \) and a step reference \( r[k] = [5, 5]^T \) applied at \( t = 5 \) s (i.e., \( r_1[k] = r_2[k] = u_s[k - 5] \)), where \( u_s[k] \) represents the unit step signal. The dashed lines in Figs. 3a and b represent the reference signal. These results show that the combined use of GSHFs and TTSM control can surmount the design limitations caused by the singularity of the ZOH matrix \( CB_d \).

The tests proposed in [24] indicate that this system has one unstable DFM at \( s = 0.01 \). This means that this system cannot be stabilised using any decentralised continuous-time LTI controller. Nevertheless, the tests proposed in [24] indicate that the DFM in this system is unstructured, and hence it can be eliminated by means of a sampled-data controller. Using a discrete-time control law with ZOHs, one can eliminate the DFM; however, the associated matrix \( G = CB_d \) is not be diagonal. For example, a sampling period of \( T_s = 1 \) s yields

\[
CB_d = \begin{bmatrix} 0.5188 & 41.2418 \\ 10.5188 & 21.1415 \end{bmatrix}
\]

It is therefore desirable to use GSHFs to solve the decentralised TTSM control problem. One can easily verify that the matrix

\[
B_d = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}
\]

results in a diagonal and invertible matrix \( G \). The farthest pole of the system (1) with the above matrices from the imaginary axis is located at \( s = -0.11 \); therefore a sampling
If the value of freedom to solve the subset of non-redundant scalar equations in (6) is given by (6b) – two first-order polynomials yield four design parameters, which provide enough degrees of freedom to solve the subset of non-redundant scalar equations in (6). If the value of $T_s$ is given, then the solutions $\phi_1(t)$ and $\phi_2(t)$ to the set of integral equations given by (6b) are uniquely defined. These unique solutions for $T_s = 1$ s are

$$\phi_1(t) = -23.89t + 11.74,$$
$$\phi_2(t) = -11.65t + 5.76$$  \hspace{1cm} (55)

The hold functions (54) satisfy the decentralisation requirements but, to reduce the inter-sample ripple effect, one can use them as the initial points in the optimisation algorithm proposed in [25]. This results in optimal second-order polynomials which satisfy the required constraints while minimising the inter-sample ripple effect. The optimisation algorithm yields the optimal second-order polynomial hold functions

$$\phi_1(t) = 1.07t^2 - 24.32t + 11.18$$
$$\phi_2(t) = -0.39t^2 - 10.58t + 5.15$$  \hspace{1cm} (56)

and the optimal sampling period $T_s = 0.9615$ s. These parameters are optimal in the sense that they minimise the expected value of the continuous-time quadratic performance index

$$J = \int_0^\infty (y^T(t)y(t) + u^T(t)u(t)) \, dt$$

over a set of uniformly distributed initial states of the system and zero initial states of the controller [25].

For the above hold functions and sampling period and for the control parameters, $\theta_1 = 0.1$, $\theta_2 = 0.3$, $\lambda_1 = 0.75$, and $\lambda_2 = 0.5$ the TTSM controller is given by

$$v_1[k + 1] = v_1[k] + 0.78\{y_1[k] - y_1[k + 1]
+ 0.0917(r_1[k] - y_1[k])\}
$$
$$v_2[k + 1] = v_2[k] + 0.52\{y_2[k] - y_2[k + 1]
+ 0.2506(r_2[k] - y_2[k])\}
$$
$$u_1(t) = 0.5547\phi_1(t)v_1[k]
$$
$$u_2(t) = 1.1518\phi_2(t)v_2[k]
$$

and is evidently decentralised because, as desired, the matrix

$$G = \begin{bmatrix} 1.7336 & 0.0000 \\ 0.0000 & 0.8348 \end{bmatrix}$$

is diagonal.

Fig. 4 gives the control signals and the output signals for the closed-loop system with non-zero initial conditions and zero reference signal $r[k] = [10, -1]^T$ applied at $t = 1$ s with the controller parameters given above.

These results show that the combined use of GSHFs and TTSM control can surmount the design limitations caused by the presence of an unstable DFM to generate a stabilising decentralised control law.

7 Conclusions

The main contribution of this work is a new constructive and simple method to design stable discrete-time decentralised control systems with TTSM. The proposed controller is LTV and can be applied to a class of continuous-time systems with coupled input–output channels, for which local LTI controllers are ineffective. The control law for each input–output agent consists of a sampler, a GSHF

Fig. 5 Response of the close-loop system to the step reference signals and zero initial conditions in Example 2

$a$ Output signal $y_1(t)$
$b$ Output signal $y_2(t)$
$c$ Control signal $u_1(t)$
$d$ Control signal $u_2(t)$
and a first-order linear compensator. One of the advantages of the proposed control structure is that it can stabilise systems with a certain type of unstable DFM, namely UDFM. This is an important benefit since no LTI decentralised controller can stabilise systems with unstable DFMs. Moreover, this control method may achieve output control of a class of plants that do not admit the conventional TTSM control method.

There are some practical issues to be taken into account. First of all, the sampling period must be sufficiently small to satisfy the requirements of TTSM control. Additionally, the choice of the GSHF for each control agent is not unique. One can find the GSHFs so that the structural requirement for decentralised TTSM control is met and, at the same time, the inter-sample ripple effect is minimised.

The combination of the TTSM controller and GSHFs conveys advantages as well as disadvantages. The main drawback of GSHFs is that they are prone to robustness difficulties in the continuous-time domain. The main advantage, on the other hand, is that the class of LTI systems to which decentralised TTSM control is applicable is significantly enlarged. In other words, the system may not have the required structure for this type of controller but, under some conditions, GSHFs can change the structure of the system in the discrete-time domain so that a discrete-time decentralised TTSM control law can be applied. Simulation results show the effectiveness of the proposed control law when the conventional LTI decentralised TTSM controller cannot be applied because of the coupling between the input–output channels and also because of an unstable UDFM in the system.

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9 References